Misconceptions leading to errors in elementary algebra (generalised arithmetic)

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MISCONCEPTIONS LEADING TO ERRORS IN ELEMENTARY ALGEBRA (GENERALISED ARITHMETIC)

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ABSTRACT

This thesis describes an investigation into the reasons underlying particular errors in elementary algebra or 'generalised arithmetic ' which an earlier project at Chelsea College - the Concepts in Secondary Mathematics and Science project - had shown were widely prevalent among 13 to 15 year old children.

The investigation comprised three phases: (1) an exploration of the conceptual bases of the errors under study, by means of a total of 72 individual interviews with children identified as making the errors; (2) the monitoring, via a series of small-group teaching experiments using three groups of 5-6 children aged 13 to 14 years, of children's interaction with a 'cognitive-instructional' programme based on the interview findings, and which aimed to help children restructure their thinking so as to avoid making the errors; and (3) the development and trial of this instructional programme for use with with whole classes, using four groups of 12 to 15 year olds taught by the researcher, and seven classes in the same age range taught by the usual class teachers. The schools used were mainly in the Greater London Area.

The results of the research indicated three main areas of difficulty contributing to the errors in question, namely (i) the way in which children view letters in algebra (as 'objects' or as specific rather than generalised number), (ii) children's difficulties in formalizing and symbolising arithmetic procedures, and in particular the use by children in arithmetic of informal 'child-methods' which do not readily extend to the algebraic case, and (iii) difficulties with

particular aspects of algebraic notation and convention (i.e. the representation of algebraic addition and the use of brackets).

In general, the teaching programme designed to help children overcome these difficulties was successful, in terms of an improved performance on relevant test items administered both immediately after the teaching, and after a two to four month delay. Variations in effectiveness of the programme with respect to the different areas of difficulty were, however, noted, and possible reasons for this occurrence put forward.

The findings from the research suggest that children's performance in elementary algebra may relate both to the child's general level of cognitive maturity, and to the kind of 'knowledge framework' with respect to arithmetic and algebra which the child has constructed. The implications of the findings for the teaching of algebra are discussed, and suggestions made for further research in this area.

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Last, but by no means least, are the many special thanks due to the teachers and children who participated so willingly in the study.

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CHAPTER 1: INTRODUCTION

This thesis describes an investigation into the causes of secondary-school children's errors in selected aspects of elementary algebra, or 'generalised arithmetic'. The investigation formed part of the Strategies and Errors in Secondary Mathematics (SESM) project, which was funded by the Social Science Research Council and based at the Centre for Science and Mathematics Education, Chelsea College, from January 1980 until December 1982. This project itself followed on from the work of the mathematics section of an earlier research programme, the Concepts in Secondary Mathematics and Science (CSMS) project, which was likewise funded by the Social Science Research Council and based at Chelsea College. In common with the area of algebra delimited for study by the CSMS (Mathematics) project, the SESM project and hence the investigation reported upon here has used the term 'algebra' in the sense of 'generalised arithmetic', by which is meant the use of letters for numbers and the consequent writing of general statements representing given arithmetic rules and operations. The algebra of solving equations, factorising, and the simplification of complex, rational, or higher order expressions has not been included in the study.

In essence, the aim of the SESM programme was to investigate the causes underlying particular errors in secondary mathematics which the earlier CSMS (Mathematics) project had shown to be widely prevalent among second to fourth year children in English secondary schools (Hart, 1980b, 1981a). Taking as a starting point the assumption that such widely occurring errors were unlikely to be the result of carelessness or casually idiosyncratic reasoning, the project adopted the view that errors of this kind were symptomatic of the operation of fundamental misconceptions on the part of those children making the errors. Consequently an investigation of the errors was expected to provide insight into the nature of these misconceptions. Such insight was considered necessary in order to best help children avoid making the errors, by indicating instructional procedures which would help children rectify the misconceptions upon which the errors were based. A second aim of the SESM project was therefore to investigate the effectiveness of short teaching modules which were based upon the information derived from the analysis of the causes underlying the errors and which aimed to help children restructure their thinking so as to avoid making the errors. The approach adopted by the team was to divide the research programme essentially into three main phases:

(1) an investigation into the causes of the errors under study by means of individual interviews with children identified as making the errors,

(2) the conducting of small-scale teaching experiments based on this analysis, and

(3) the development of prototype teaching modules for trial with whole classes.

The topic of elementary algebra was considered to be a particularly important one for this kind of analysis. That children have considerable difficulty with this area of mathematics has been highlighted by the results of both the CSMS (Mathematics) research (see Küchemann, 1978, 1980, 1981) and the Assessment of Performance Unit's secondary surveys (1980, 1981, 1982), thus reinforcing what teachers of algebra have long observed. The Department of Education and Science report on "Aspects of Secondary Education in England" in 1980 noted that "the teaching of traditional algebra has long presented difficulties in schools" and that "the time has come for a careful reappraisal of the aims and content of algebra courses, and of ways of teaching the subject" (p.128), a point also reiterated more recently by the Committee of Inquiry into the Teaching of Mathematics (1982).

Such a reappraisal itself requires a careful consideration of children's capabilities and needs in this area. Certainly children's ability to work with understanding in any aspect of algebra must depend upon a firm grasp of those concepts most fundamental to the topic. In this regard, the notion of 'variable' assumes particular importance. Not only is this notion "fundamental to algebra and all higher mathematics (Wagner, 1981b, p.107), but an operational understanding of the concept is important to work in other subjects such as science, and to a constructive understanding of computer usage. It was therefore thought especially useful to investigate more closely some of the specific difficulties which children have concerning the meaning of letters and the uses to which letters are put. By virtue of its focus on just these issues, the work of the CSMS team in generalised arithmetic provided a particularly valuable starting point.

As a result of the approach taken by the present study, it was anticipated that the products of the research described in this thesis would be of two kinds. Firstly, it was hoped to provide an improved understanding of the difficulties which children have in attempting to learn algebra, and of the misconceptions and erroneous procedures to which those difficulties might relate. Secondly, it was hoped to produce an actual teaching programme designed to alleviate some of these difficulties and which could be used by teachers as it stands, but which would preferably form the basis for future curriculum development in this topic. In addition, it was hoped to use the results of the investigation to reflect upon the status of some current theories of cognition as they might apply to the learning of algebra, and to make suggestions for further research in this area.

The thesis is divided into three main parts. The first of these looks at the background of the study, considering such aspects as the notions of 'error' and 'strategy' to be adopted, the psychological and methodological framework appropriate to the investigation, and the contributions of other research on children's understanding of algebra. The second part describes the research programme itself, presenting first the 'functional analysis' or interview phase and then the teaching experiments and class trials. Part three discusses the findings in the light of more recent research evidence, and considers the implications of the findings for teaching and further research, and for the development of a psychological model for children's learning of mathematics in general.

PART I

BACKGROUND TO THE STUDY

CHAPTER 2: THE DEVELOPMENT OF THE SESM PROJECT

General Background

Prior to the work of the CSMS (Mathematics) project, there had been relatively little research in England which had specifically addressed the question of the kinds of mathematical concepts which children acquire at different stages of secondary school study. The curricular and organisational reforms of the 1960's and 1970's had resulted in tremendous changes in curriculum content and teaching approaches in secondary school mathematics, and in a widening of the ability range to which the subject was taught. There had been, however, little research upon which to base these changes, and little guidance from the psychological literature to permit a clear description of the kinds of mathematical activity and concepts that were likely to be understood by children at different stages of secondary schooling. At primary school level the work of Piaget had had a profound effect on teaching methods and curriculum content in mathematics, and had focussed educators' attention more sharply on the need to match instruction to the cognitive level of the child. A 'concept map' in primary school mathematics had even been drawn up on the basis of the kind of cognitive abilities which Piaget had indicated that children of primary school age acquire (Nuffield Mathematics Project, 1970). Teachers who acknowledged the same need to match instruction to children's cognitive functioning at secondary school level, however, looked in vain for research or theory to guide them. The CSMS (Mathematics) project attempted to help remedy this situation by endeavouring, among other aims:

"1. to identify order of difficulty throughout the treatment of individual topics in currently developed courses in... mathematics and to formulate and test hypotheses concerning the difficulties;

2. to extend the concept 'map' to secondary level and to indicate probable outcomes of different partially-ordered teaching sequences within its framework;

3. to provide evaluatory procedures designed to help teachers to identify the stages reached in their pupils' thinking and which would also stand up to external scrutiny."

(CSMS Research Proposal, 1973)

The Work of the CSMS Project

By administering separate paper and pencil tests of understanding in different topic areas of mathematics (such as decimals, fractions, ratio, algebra, graphs and so on) to large representative samples of secondary school children aged from 11 to 16 years, the CSMS (Mathematics) team aimed to delineate a hierarchy of levels of understanding of concepts in each area, and to characterise the levels of the hierarchy so established by cognitive or mathematical descriptors. The tests of understanding were developed on the basis of the concepts in each topic which the researchers thought important, and on the basis of textbook analyses, discussions with teachers, psychological considerations and reference to the relevant research literature, and most importantly, upon the findings from individual interviews with children. These interviews provided information which was useful in determining the format and language of the test items, but even more importantly, gave insight into the kinds of concepts which children actually have, and the kinds of procedures that they employ in solving different mathematical problems. In the case of the research on algebra, 27 children aged from 13 to 15 years were interviewed, and a total of 3550 children in the same age range received the final written test.

In essence, the results of the CSMS research suggested that a hierarchy of levels of understanding could be identified within each topic area studied (Hart, 1980b, 1981a). This 'hierarchy' was formed as the result of a statistical procedure based upon an initial ranking of the items of a given test in terms of facility, followed by a grouping of selected items within that facility array in such a manner that individual children who correctly answered a specified proportion (two-thirds or more) of a 'higher' (harder) grouping of items had also correctly answered this proportion of items in each 'lower' (easier) grouping (Figure 2.1; see Hart, 1980b; Hart & McCartney, 1979 for details of the statistical procedure). By examining the items in each grouping it was possible to describe 'levels of understanding' according to the concepts and procedures upon which the items were based. The number of levels of understanding identified in this way varied according to the topic, ranging from three for graphs to seven for vectors. In order to permit comparison across the different topic areas, these levels were subsumed into four main 'stages', of which stage one was the lowest and stage four the highest stage of understanding. Consideration of the number of children operating at each stage showed that there was a wide variation in understanding demonstrated by children of the same age, that items representing the highest stage of understanding were answered successfully by a surprisingly small percentage of children (given that items of this kind are typically taught in all secondary mathematics courses), and that in many cases the child's understanding as defined by the CSMS hierarchy improved relatively little as the child progressed from

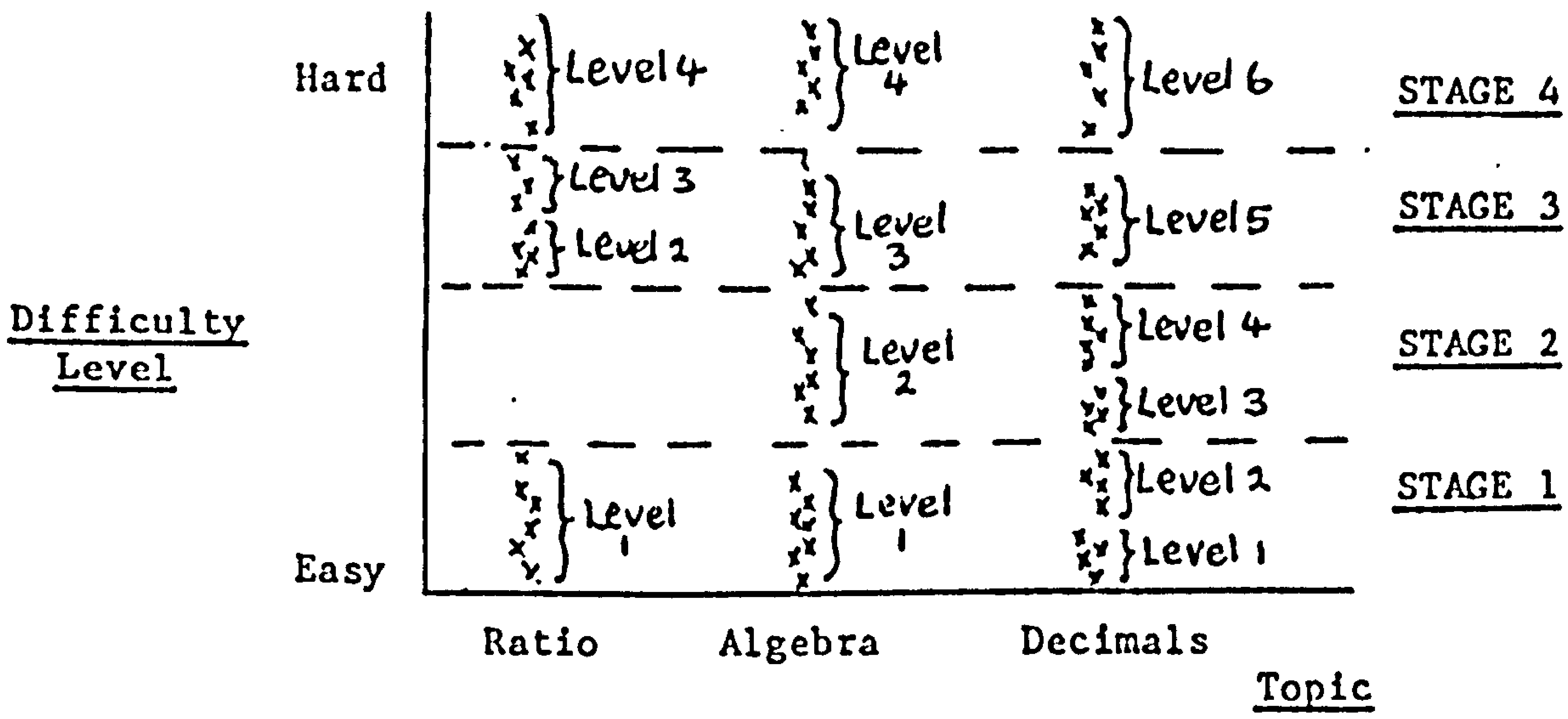


Figure 2.1 Diagram illustrating formation of CSMS hierarchy by ranking items (shown by crosses) in each test in order of facility, followed by statistical grouping into 'levels of understanding' for each topic. The array thus formed is then divided into four 'stages of understanding' in order to allow comparison between topics.

second year (age 13) to fourth year (age 15) (see Chapter 4 for details of these results in algebra). In addition, the results revealed certain commonly-made errors in secondary mathematics, sometimes being observed in 40 per cent or more of the children tested, and which appeared to signal the operation of wide-spread basic misconceptions among children in this age range. Findings from the interviews conducted as part of the research programme had indicated that some of these misconceptions were associated with the use by the child of naive, non-taught procedures in mathematics. However, more information on this issue was needed.

The Aims of the SESM Project

Of perhaps even more interest than the kind of question that children solve correctly at each level is the nature of the errors they make, especially when the same kind of error is made by large numbers of children. The study of these errors is particularly important because of the information it provides concerning the ways in which the child views the particular problem, and the kinds of concepts and strategies that are used in attempting to solve the problem. The value of studying children's errors, therefore, lies in the insight which such study can provide into the child's underlying cognitive processes. This information is of interest not only because it might help to suggest ways of encouraging children to avoid making these particular kinds of error by helping them to first restructure their thinking in this regard, but also because it might explain secondary school children's apparently slow progress in attaining higher levels of understanding in mathematics, by giving a clearer picture of the cognitive mechanisms by which children's actual mathematical performance is determined. Consequently the SESM project set out to investigate this issue by analysing the reasons for those errors which occurred most frequently, and by investigating the effectiveness of teaching programmes designed to help children avoid the misconceptions thus identified, or to correct them if already formed. The stated aims of the project were therefore:

"1. to investigate in depth the reasons why children commit the errors already identified in the wide scale CSMS survey;

2. to ascertain the connection between the types of errors committed and the general level of mathematical understanding exhibited by the child (in terms of the CSMS hierarchy);

3. to investigate selected methods of improving the strategies and preventing the errors."

(SESM Research Proposal, 1979)

It was intended that the investigation be carried out primarily by means of individual interviews with children identified as making the errors in question, followed by the development of teaching programmes based upon the findings derived from these interviews and which would then be tested with small groups and eventually whole classes of children.

The Notion of 'Error'

Overview

The interest in children's errors in mathematics shown by SESM is, of course, by no means new. There exists a substantial volume of research into children's errors in computation in particular, and also on the errors children make in attempting to solve arithmetic word problems, ranging from early class-based research (e.g. Brownell & Moser, 1949; Uhl, 1917, in Radatz, 1980), and analysis of written scripts (Arthur, 1950; Cox, 1975; Knifong & Holtan, 1976; Roberts, 1968) to investigation via individual interviews with children (Erlwanger, 1973; Ginsburg, 1977a; Lankford, 1974), and computer assisted research (Brown & Burton, 1978, Brown & Van Lehn, 1980, 1981). While some error studies have been concerned with using the information gained to improve instruction (e.g Ashlock, 1976; McAloon, 1979; Reisman, 1980) or as a positive stimulus to investigate new areas of mathematics (Meyerson, 1976), much of the research has concentrated on the definition of error categories and the distribution of errors (e.g. Brueckner, 1935; Buswell & Judd, 1925; Englehardt, 1977; Grossnickle, 1939; Roberts, 1968). By this approach, attention was focussed on the errors themselves, and while some errors at least were recognised as surface manifestations of consistently applied misconceptions, in the main researchers stopped short of attempting to analyse the thought processes which led to the errors. Consequently, few attempted to relate their findings to a particular theory of cognition (although Radatz, 1979, noted the need to complement research on error patterns with cognitive models of the causes of errors) and few were able to suggest reasons for the development of the misconceptions. More recently, a growing number of researchers have been less concerned with classifying errors and more concerned with using error analysis as a means for investigating basic structures in the mathematical teaching-learning process (e.g. Davis, 1975a, b; Davis, Jockusch & McKnight, 1978; Davis & McKnight, 1979, 1980; Erlwanger, 1973, 1975; Ginsburg, 1975, 1977a, b). Similarly Reisman (Reisman, 1980; Reisman & Kauffman, 1980) has suggested that children's errors in mathematics often arise as the result of a teaching sequence which does not take adequate account of the psychological (as opposed to logical) prerequisites of a given learning task, giving several examples from primary school mathematics to show how a discrepancy between the traditional approach to teaching these topics and the psychological complexities of the task involved can lead to confusion for the child. For Reisman, error analysis (accomplished by task analysis, observation, and individual interview) is essential as a first step towards improved curriculum design aimed at minimising such errors.

This latter approach, by which children's errors are related to their thinking processes, owes much to the Piagetian tradition, not only in terms of methodology used, but also in terms of its basic assumption that errors are symptomatic of a particular way of reasoning. Piaget drew attenion to the fact that an analysis of children's errors can often reveal as much information about the child's reasoning processes as can an analysis of the child's correct responses. He also drew attention to the similarity in erroneous answers given by different children of comparable age, and the differences which appeared between successive age groups. Consequently he used these apparent errors in reasoning demonstrated by children at different ages in order to elaborate his theory of cognitive development marked by a series of stages of cognitive 'structure', each of which permitted certain kinds of logic but denied the child access to others. The errors which children made thus provided insight into their actual ways of thinking, and into the kind of psychological apparatus which underlies those ways of thinking and which is common to children of a given age (Piaget, 1926a, b; 1950, 1953; Piaget & Inhelder, 1964).

The variation in research interests and methodology indicated by the above discussion both reflects and results in a variation in what is implied by the use of the term 'error'. Researchers concerned with the identification of error types, for example, typically give equal regard to non-systematic errors due to such occurrences as misreading a question or guessing a solution, and to systematic errors, whether these be the result of wrong or misleading information which the child has been given, or of misconceptions or incorrect procedures which the child has constructed. Those who use errors as a means of studying children's reasoning processes, however, concentrate attention on systematic errors founded on basic misconceptions which the child consistently applies. Even here, a difference in approach can be distinguished between those (e.g. Davis, Erlwanger, Ginsburg) who stress the individual nature of errors, considering these to be due to very individual problem-solving strategies and rules which go back to a given child's earlier experience and understanding, and researchers such as Piaget, whose work focusses on those errors which are common to different children, and which are therefore suggested to be indicative of general ways of reasoning characteristic of the given population at large. This distinction, and its possible implications, will be referred to again at a later point in the thesis.

A Definition of 'Error'

In view of these different emphases and perspectives in the analysis of errors, it is useful to specify at the outset where the present study stands in this regard. As will be clear from the aims of the SESM project described earlier, the view taken by the project, and hence by the research described here, is that an 'error' may be regarded as reflecting a consistent way of viewing a situation or of handling a problem's solution which is consonant with the child's current cognitive apparatus. Analysis of the child's error therefore gives insight into the child's cognitive functioning and how the child fails to cope correctly with the particular kind of problem. In this regard, 'errors' are to be distinguished from wrong answers due to guessing, misreading, or mis-information, and from mistakes resulting from miscalculations of a non-systematic nature, or from idiosyncratic responses due to other kinds of carelessness and which may be readily rectified by the child when attention is drawn to the occurrence. In order to merit such a status, it is necessary that an 'error' should fulfil certain criteria. Thus to be classified as an 'error', as defined, a particular incorrect response should be:

1. consistent, i.e. occur in at least other items of the same type, so that it is not readily attributable to the features of a single isolated item;
2. stable, i.e. made by the child over a given period of time;
3. resistant to casual re-education.

Errors which meet these criteria may be regarded as manifestations of particular ways of cognitive functioning with regard to the kind of problem under study. It is only by its regularity of occurrence that an error can be interpreted as indicating that a child is systematically attempting to use what he or she knows in a consistent application of the procedures which that child has constructed. In addition, the errors chosen for study by SESM were:

1. wide-spread, i.e. made by large numbers of children;
2. course-independent, i.e. not readily attributable solely to a particular syllabus, text, teaching method or teaching style.

This suggests that the errors in question signify a communality in cognition among children which arises despite apparent variations in learning experience. Whether this is due to the operation of

Piagetian-like cognitive stage structures, or to some other construct, or whether indeed the same wrong answers can be attributed to a range of different perceptions and procedures, is left open at this point.

Implications of the Definition

One implication of a definition of error as reflecting the child's cognitive functioning is that unlike a 'mistake', an 'error' as defined is an error only to the teacher or investigator. In so far as it is the outcome of a particular way of reasoning, it may not be considered to be an error by the child. Any subsequent attempts to develop teaching programmes aimed at the removal of such errors will therefore almost certainly need to address themselves to this point concerning the child's likely non-recognition of the limitations of his or her current ways of reasoning, if they are to achieve success. By adopting an interpretation of errors as marking a way of cognitive functioning in this way, and by its commitment to the development of teaching programmes aimed specifically at the avoidance or correction of such errors, the SESM project committed itself to a consideration of ways in which this recognition of error might be effected.

The Notion of 'Strategy'

Overview

Error of the kind described can arise as the result of what a child 'knows' or as a result of the kinds of procedure which he or she constructs; the misconception which forms the basis of the observed error may lie in the child's conceptual or knowledge store or in the strategies which are developed in order to handle the problems under study. These 'strategies' may be general learning or problem-solving approaches characteristic of a given individual and hence indicative of that individual's cognitive structure and/or personality, or they may be related to the kind of problem under study and hence task rather than individual-dependent. In his work on learning strategies, Pask (1976) has attempted to distinguish between responses to the requirements of a particular kind of task which he terms 'strategies', and learning 'styles', which he sees as being a relatively stable preference of an individual for a particular way of learning (see also Entwistle, 1978). Nevertheless, the 'strategies' which Pask describes appear to be strongly related to the styles considered to be characteristic of the individuals employing them, so that it is not easy to clearly differentiate between the two, despite Pask's stated intention that this be done. Other workers concerned with learning strategies have stressed both the consistency in approach to learning-tasks shown by given individuals (e.g. Svensson, 1977), and the variation in approach according to the task involved (e.g. Marton and Säljö, 1976a, b; Laurillard, 1978, quoted in Entwistle, 1979). Thus Marton and Säljö (1976b) have emphasised that the strategy used by a given individual depends critically on the content and context of what is learned, as does Entwistle in his reminder that "the verb 'to learn' takes the accusative" (Entwistle, 1976).

While the content and context of what is learned may affect the strategy an individual adopts on a particular occasion, it seems clear that there may also be observed consistencies in approach which may more appropriately be attributed to the general mode of cognitive functioning characteristic of the individual. As a result of a series of investigations into children's approaches to the solution of specially designed tasks, for example, Bruner and his co-workers (Bruner, Olver & Greenfield, 1966) were able to distinguish a progression of problem-solving strategies from 'trial and error', through 'systematic trial and error' and 'successive pattern matching' to an 'information selection' approach. Not only was a given individual observed to use a particular strategy on different tasks, but the same strategy was also observed to be used by other children of comparable age. While this progression was 'developmental' in the sense of strategies which were observed to occur spontaneously in an order which corresponded to successive age groups, there was no suggestion that the appearance of a particular strategy was 'fixed' to a given stage of development, as was the case, for example, for Piaget's cognitive structures. Indeed, Olson (1966) took care to show that there can be a great difference between what the child spontaneously does and what the child is capable of doing once the existence of a superior strategy is known to him. Working within a Brunerian framework, Olson presented seven-year-old children with a task which required them to identify which of two given patterns (e.g. T or L) was 'correct' in the sense of having been pre-programmed on a board equipped with rows of small push-buttons. If all the buttons were pressed, those that formed the selected pattern would light up; the rest would remain unchanged. The child's task was to find which pattern was 'hidden' in this way on the board. 90 per cent of the seven year olds tested spontaneously adopted the successive pattern matching strategy by which each pattern was tried in turn, with all the buttons relating to the first pattern being depressed before the second pattern was investigated, again in its entirety. However, when Olson included a 'constrained' condition in which the child was asked after each button-press such questions as 'Do you know now which pattern is correct?' and 'Can you find out with just one more button-press?', the results were quite dramatic, with 80 per cent of the seven year olds now moving onto the information selection approach characteristic of older children and adults.

Thus while the particular cognitive structure or system that a child 'naturally' works within at any given time may promote a particular kind of strategic approach to solving problems, it is by no means certain that other more sophisticated strategies are necessarily unavailable to the child at that stage. Indeed, it may even be suggested that it is by becoming exposed to more advanced strategies that the child's cognitive system is stimulated to undergo a restructuring which may pave the way toward the next level of cognitive functioning. Certainly there exists evidence to show that concrete and formal operational thinkers differ in their use of such strategies as random trial and error, systematic trial and error, and use of 'deductive algorithm' (Days, Wheatley & Kulm, 1979), indicating that strategic approach and cognitive level are linked. The apparent ability of individuals to improve their level of cognitive functioning after strategy instruction has also been noted elsewhere (e.g. Lawson & Kirby, 1981; Lawson & Wollman, 1976; Rosenthal, 1975) and forms the basis of the cognitive enrichment programme developed by Feuerstein (see for example, Feuerstein, Rand, Hoffman & Miller, 1980). One is tempted to suggest that any hopes of improving cognitive functioning may rest on a two-fold approach by which the child is both encouraged to perceive the limitations of his or her current way of reasoning (i.e. to recognise existing 'errors' in conception or reasoning), and to investigate the superiority of more advanced kinds of strategy. Certainly attempts to promote cognitive acceleration by other considerations have proved equivocal (e.g. Danner & Day, 1977; Shyers & Cox, 1978).

A Definition of 'Strategy'

With this possibility in mind, the importance of studying children's strategies and errors in any given area of intellectual activity assumes greater proportions. By 'strategy' will be meant any generalisable approach that constitutes a plan for executing selected procedures aimed at the solution of a given problem. By contrast, the term 'procedure' will be used to refer to the particular sequence of operations which is selected to solve the particular problem under consideration, or other problems of the same type. For example, 'looking for an appropriate algorithm' constitutes a strategy; 'multiplying length by width to obtain a measure of area' represents a procedure. Since strategies are thus defined as generalisable approaches appropriate to a wide range of problems, it would be expected that some consistencies in strategy-use might be observed both within and across individuals. As in the case of the cognitive factors underlying 'error', however, the question as to whether the kind of strategy used by a child is a consequence of some particular cognitive 'stage' structure, or whether it is more dependent upon some alternative structure related to the content and contextual features of the task itself, is left unspecified at this point.

The Nature of Mathematical Strategies

In defining what will be meant by 'strategy' in the present study, it is useful in addition to indicate the kinds of strategy that might be observed within the context of answering mathematical questions. While the kinds of strategy appropriate to mathematical activity have not been fully researched, the work of such researchers as Krutetskii (1976), Karplus (Karplus, Pulos & Stage, 1981), and Bruner and his colleagues (Bruner, Olver & Greenfield, 1966) provides a base from which a range of suggested strategies might be described. While Krutetskii's work was more particularly concerned with specifying the components of mathematical ability, it was apparent that the possession of 'mathematical ability' was related to the kind of approach adopted in solving mathematical questions. Thus Krutetskii noted that:

> "significant qualitative differences were observed on the nature of the trials by capable and incapable pupils. The trials made by mathematically inept pupils always bore the character of blind unmotivated manipulations, chaotic and unsystematic attempts to find a solution (more accurately, they were attempts at guessing, at coming across a solution at random).
>
> Capable pupils, however, were marked by an organised system of searching, subordinated to a definite program or plan."
>
> (Krutetskii, 1976, p.292)

In particular, one of the basic characteristics of mathematical thought was described as:

> "an ability to formalize mathematical material, to isolate form from content, to abstract oneself from concrete numerical relationships and spatial forms, and to operate with formal structure"
>
> (Ibid, p.87)

The recognition of formal structure and consequent selection of an appropriate mathematical procedure might therefore be suggested to be an important strategy in mathematical problem solving.

While Karplus is perhaps not especially associated with the notion of strategies in mathematics, his 'reasoning patterns' might in fact be more appropriately termed strategies, providing a list which includes unrelated trials, trial and error cycles progressing systematically, and (in the context of 'number puzzle' problems) forming an equation and solving it (in this case a particular procedure which is an example of the strategy of recognising formal structure) (Karplus, Pulos & Stage, 1981). Similarly the definitive work on strategies by Bruner and his co-workers has already been referenced, and indicates a range of strategies including random trial and error, systematic trial and error, successive pattern matching, and information selection (Bruner, Olver & Greenfield, 1966). Whilst not developed specifically within the area of mathematics, this latter analysis appears to have considerable applicability to this subject, and forms a useful basis for the elaboration of a more complete list of strategies than that provided by Krutetskii's or Karplus' work.

Starting from the above mentioned research, therefore, it is suggested that the following analysis describes a plausible range of strategies that might usefully be explored in reseach on children's mathematical activity:

RTE - random trial and error,

STE - systematic trial and error,

IS - informal ('intuitive') strategy,

SP - search for pattern (e.g. in numerical data),

SA - search for algorithm (i.e. taught rule),

RFS - recognition of formal structure of problem (and consequent selection of appropriate procedure).

In its investigation of children's errors and strategies, therefore, the present study set out to investigate the possible operation of strategies of the above kind.

CHAPTER 3: PSYCHOLOGICAL AND METHODOLOGICAL PERSPECTIVES

PSYCHOLOGICAL PERSPECTIVES

Whilst a specific theoretical framework had not been rigorously decided upon for the present research, it was intended to work broadly within a Piagetian framework. However, mindful of the criticisms levelled against the Piagetian theory, particularly in terms of such issues as the invariance of the 'stage' construct across different tasks and contexts (e.g. Brown & Desforges, 1977; Flavell, 1982; Smedslund, 1977), the apparent uncertainty concerning the age at which children might progress from one stage to another (see Dale, 1970; Lovell, 1961a, b; Lunzer, 1965, 1970; McNally, 1970; and Shayer, Küchemann and Wylam, 1976; with regard to the age of concrete to formal operational transition) and the concern over the validity of the formal operational stage description (e.g. Bynum, Thomas & Weitz, 1972), it was considered inappropriate to adopt the theory in toto as the guiding paradigm for research into children's understanding of mathematics in the 12 to 16 years age range.

The problem concerning the age at which transition from concrete to formal operational thinking occurs is, of course, readily met by allowing for the possibility of either mode of reasoning to be demonstrated by children in the age range studied (namely 12 to 16 years). The criticism concerning the validity of Piaget's description of formal operational thinking is of greater concern, in that the usefulness of any attempts to explain or predict children's mathematical performance in terms of their general cognitive capabilities requires that the model adopted for the latter description be accurate. However, perhaps of most fundamental importance is the criticism concerning the invariance of the stage construct, in that this reflects upon one of the central tenets of the Piagetian theory.

The Piagetian Model

Overview

Piaget's model was based upon the notion that despite the possible variety of patterns of intellectual behaviour which may be observed in any individual, there is underlying them a specific way of reasoning which explains these behaviours and gives rise to a unity of consistency in the individual's intellectual approach. This mode of reasoning was not to be thought of as fixed, however, but as undergoing change through a series of 'stages' in a predetermined manner and as the consequence of maturation and experience. Piaget argued (Piaget, 1950) that each mode of reasoning was the outcome of a particular kind of 'cognitive structure', or way of organising experience and dealing with the environment, which defined the limits of the child's intellectual functioning and so determined the kind of logical thinking available to the child and hence the kinds of task which could be successfully handled. Progression beyond these limits must await a re-organisation of this structure so as to permit ways of reasoning which had not been possible before. The differences in ways of reasoning attending successive structures were therefore differences in kind and not merely of degree, so that while adult forms of reasoning developed out of the child's, the two were qualitatively different and admitted a different kind of logic (Piaget, 1953; see also Beilin, 1971). The same kind of logic would, of course, be applied to all intellectual activities exhibited at any given stage.

Criticisms of the Piagetian Model

Piaget's theory thus looked to the operation of general capabilities which develop through a series of stages but which are largely independent of the specific contents or contexts of the tasks to which they are applied. This notion has been attacked on several (related) grounds:

(1) there is no evidence for a single unified structure at any stage,

(2) children may show reasoning of a kind generally taken as representative of a much later stage,

(3) children may show an absence of expected logical operation, and

(4) there is evidence for task specificity in behaviour.

Evidence for a unified stage structure. One of the most widely quoted criticisms in this regard is that of Brown & Desforges (1977; see also 1979). Whilst accepting that at times of transition from one stage to another there may be some variation in appearance of operations characteristic of each stage, Brown and Desforges stress that there must be 'a substantial degree of homogeneity' (1977, p.9) among behaviours manifested by children for the greater part of each stage if the notion of 'stage' is to have any validity (see also Novak, 1977). From a consideration of various studies reporting correlational data for stage-linked behaviours, Brown and Desforges conclude that such a homogeneity is in fact not observed, and that the stage concept is therefore untenable. Shayer (1979) has criticised this conclusion, arguing that not only were Brown and Desforges highly selective in their choice of correlational data and that other studies report substantially higher correlations (e.g. Bart, 1971; Lee, 1971; Lovell, 1961a, each cited in Shayer, 1979), but also that the values quoted by Brown and Desforges are themselves found in many cases to be of a substantial order when adjusted to compensate for restricted range of measures. Nevertheless, even these 'higher' correlations, including those reported for Shayer's own data, are generally of the order of 0.6, leaving open the question as to whether this value is to be regarded as high enough to support the notion of stage unity or 'structure d'ensemble'. Certainly correlations of this magnitude, consistently reported, indicate some communality of functioning; however, it would seem that the assignment of a given cognitive developmental stage provides only a first approximation of children's abilities on different tasks. A reasonable proportion of the variance is still left unaccounted for, a proportion which may be explained by factors specific to the different tasks involved, thus perhaps providing some evidence for the view that task-related factors are also important in determining a child's level of response (see point (3) above, also the discussion later in this chapter). However, the value of using correlational data from Piagetian studies in order to assess the utility of the stage construct is in any case open to doubt. The use of the Piagetian clinical interview has typically resulted in small sample sizes, the criteria for judging level of response in Piagetian terms have often not been rigorously specified, and disagreement has existed perhaps as a consequence between different researchers as to the level at which to categorise various behaviours. In addition, the magnitude of correlation values can be markedly affected if the range of measures is very narrow. In view of this, the wide range of results indicated by Blasi and Heoffel's review (1974) of studies on formal operational reasoning may reflect procedural differences or sampling variation as much as heterogeneity of performance, as Shayer has pointed out (1979, p.272). In either event, the evidence for the idea of unified stages appears to be mixed (see Flavell, 1982). Hence some researchers continue to remark upon the apparent differences in behaviour shown by children on tasks of comparable structure:

> "All kinds of discrepancies crop up with children of all ages and with adults, and with all kinds of concepts and structures. A child behaves in one way in one situation and in another way in another situation which may appear strictly equivalent to the first situation as far as task structure is concerned."
>
> (Smedslund, 1977, p.2)

whilst others seem compelled to retain the stage concept even if "like a spectre it eludes capture" (in Driver, 1978, p.58).

Precocious demonstrations of higher-stage reasoning. A number of studies have shown that, by varying the features of a given Piagetian task (e.g. Bryant & Tabasso, 1971, but see Halford and Galloway, 1977, for an alternative interpretation of these findings; Gelman, 1972; Povey & Hill, 1975), by altering the form of the questions asked (Donaldson, 1978), or by the intervention of minimal training procedures (e.g. Brainerd, 1974, 1978; Gruen, 1965; Hamel & Riksen, 1973; Rosenthal, 1975), children have been observed to demonstrate attainment of certain logical operations at a much younger age than that specified by Piaget. Such observations do not, of course, dispute the existence of the operations described by Piaget. Indeed, they acknowledge their appearance, albeit at a much younger age than indicated by the Piagetian analysis. However, nor do they merely contest the Piagetian age-stage relationship. Rather they present an attack on the validity of the Piagetian tasks to determine whether or not a given operation is indicated, and hence upon the whole argument that operations or abilities appear in the specified clusters and fixed sequence described in the Piagetian theory. Donaldson (1978), for example, considers that children's understanding of and response to the questions posed in the Piagetian task situation depends upon their 'sense of the situation', which is determined by such factors as their view of the experimenter's intent and by the way in which they use and interpret language (the recognition that children and adults may use language differently presumably also requires explanation). Smedslund has also commented upon the question of the child's understanding (Smedslund, 1977), pointing out that there is a circular relation in the Piagetian interview situation between logicality and understanding:

> "In order to decide whether a child is behaving logically or not one must take it for granted that he has correctly understood all instructions and terms involved. On the other hand, in order to decide whether or not a child has correctly understood a given term or instruction, one must take for granted that the child is behaving logically ..."
>
> (Smedslund, 1977, p.3)

Smedslund concludes that rather than assume that the child has understood appropriately and then study the child's logicality, it makes more sense to assume the child's logicality and then investigate the nature of the child's understanding. Such an approach would presumably also reveal differences between child and adult, however, differences which it may well be that the Piagetian theory is already attempting to explain.

Hence while it can be demonstrated that a greater attention to the child's understanding of a given task (and the subsequent modification of task or instructions in line with this understanding) may result in the demonstration of more mature ways of reasoning on the part of the child, it would seem that this very difference in the nature of understanding between child and adult does itself require explanation.

In a similar way, the findings from studies demonstrating an improved level of response after training intervention are also generally less conclusive in their criticism of the Piagetian position than might initially be supposed (see Kuhn, 1974). While Beilin, for example, was able to obtain improved performance on number and length conservation (Beilin, 1965) and the representation of horizontal water levels (Beilin, Kagan & Rabinowitz, 1966), these findings have been criticised as being attributable to stimulus/response generalisation rather than the acquisition of 'true' operational structures. Assessing the value of studies which show, or of course, which do not show, enhanced performance after training, requires a careful analysis of the training which is involved. The same kind of criticism has been levelled against other training studies, such as Piaget's (1964) comments on Smedslund's (1961, 1962) 'successful' training for weight conservation but not transitivity, Inhelder and Sinclair's (1969) criticism of Kohnstamm's (1963) demonstration of class-inclusion training, and comments attributing the success of studies cited by Brainerd (1978) to children's overlearning of specific rules or strategies (see Küchemann, 1980; Pascual-Leone, 1976). The demonstration that improved performance after training results from the development of more complex structures rather than reflecting a learned response requires that attention be paid, among other issues, to the explanations which children give for their new performances (so that children are not, for example, merely parrotting explanations given during training), and that some demonstrations of transfer to associated tasks be made. As regards the first point, it seems that such information is often not reported (Küchemann, 1980; but see e.g. Hamel & Riksen, 1973; Sheppard, 1974). The issues concerning transfer of training, or generalisation, is more problematic because of the necessity to decide what kinds of transfer should be required in order to allow for training to have been successful in promoting cognitive structuring (see Anthony, 1977). Gruen (1965), for example, trained children in number conservation and found transfer to the same task with different objects, and also some evidence of transfer to conservation of length and substance. Hamel and Riksen (1973) found transfer from conservation of continuous and discontinuous quantity to different dimensions, namely two-dimensional space, substance and weight. Brainerd (1974) also found some transfer between dimensions (from length to weight) with respect to different tasks of transitivity, conservation and class-inclusion, but no transfer between these three different kinds of task. The above mentioned studies all relate to the acquisition of concrete operational tasks. The same picture is obtained where training for formal operational performance is concerned. Hence Danner and Day (1977) and Siegal and Leaver (1973) found improved performance on the pendulum task after training, but it was not clear that this reflected anything more than the result of specific response or strategy training, although Danner and Day did find some evidence of transfer between the pendulum and the bending rods task (both requiring 'control of variables'). Transfer between these tasks was also observed by Lawson and Wollman (1976) but this transfer did not extend to tasks involving other formal operational schemas such as proportionality. Shyers and Cox (1978) found no evidence of transfer either within a schema (proportionality) or between schemas (however, the nature of the 'training' used in this study is open to criticism). On the other hand, Kuhn and Angelev (1976) and Rosenthal (1975) found improvement on combinational problems and control of variables tasks respectively which could not be attributed to direct algorithmic training and which appeared to be associated with transfer effects to other tasks. According to the 'structure d'ensemble' notion of the stage construct, one might expect transfer between schemas, since if training has been successful in restructuring cognition after the Piagetian model, then all the stage-related kinds of logic should presumably appear together. However, where the concrete operational stage is concerned, Piaget himself has written that the "logical forms are not yet independent of their content. They are a structurisation of the particular content and there is no necessary generalisation" (Piaget & Inhelder, 1969, p.170), so that the issue concerning transfer of this kind is not clear cut, at least where concrete operationality is concerned. Since it is in any case accepted that at times of transition there is likely to be some variation in appearance of operations characteristic of the new stage (Brown and Desforges, 1977), it would not be expected that training programmes conducted with children in or near to the transitional phase would necessarily induce the simultaneous appearance of all such operations. Since Inhelder, Sinclair and Bovet (1974) have suggested, as the result of a series of 'naturalistic' training studies involving cognitive conflict, that successful enhancement of the child's level of functioning depends upon the child already being near the transitional phase (see also Sheppard, 1974), this consideration would mitigate against the expectation of necessary transfer of function even when transfer is to the formal operational stage, where the same restriction concerning content does not apply. "Until the generalisation question is satisfactorily answered, then pronouncing any given training study a success (or not) remains equivocal."

(Kuhn, 1974, p.593). From the above discussion, it would seem that the resolving of this issue is itself problematic.

Nevertheless, the finding that young children can, under circumstances different to those of the Piagetian tasks, or as the result of training, exhibit reasoning patterns characteristic of later stages, suggests that the notion of non-availability at any point of various logical operations be questioned. Hence Harris (1975) concluded from his study of the abilities of young children (aged 5 to 7 years) to infer the attributes of nonsense concepts from knowledge of the class membership of the concept, that the difference between 'pre-formal' and 'formal operational' thinkers may lie in the "spontaneous deployment of rules of inference rather than their availability as such" (p.151). The similar finding by Olson (Bruner, Olver & Greenfield, 1966) concerning the availability, albeit non-spontaneous, in young children of higher order strategies, has already been discussed in Chapter 2.

This observation suggests both that children may be capable of using 'more advanced' logical operations, and that they may not spontaneously do so. The argument with the Piagetian analysis thus becomes not so much one of validity of description, but rather one of what it is that is being described. The Piagetian theory, it would seem, may accurately be depicting what children 'normally' do, rather than what they may be capable of doing under optimal circumstances. Analysis of the factors contributing to the child's attainment of such 'optimal' performance may provide useful information concerning the nature of children's cognition. Gelman's study (1972), for example, suggests some interesting ideas in this regard. Contrary to the expectations of the Piagetian conservation paradigm, Gelman found that very young children (aged 6 years down to 3 years) possessed the capacity to treat number as invariant. Why, then, do children at this age typically not conserve? On the basis of the results obtained in her own study, Gelman suggested that rather than being a test purely of logical capacity, the conservation task in fact evaluates a variety of factors:

> "Put another way, it seems that children who pass the conservation test are demonstrating many extralogical skills as well as their logical capacity; whereas children who fail are doing so for any number of reasons .... Thus it seems that the conservation task is, at a minimum, a test for logical capacity, the control of attention, correct semantics, and estimation skills. Hence the ability to conserve represents a sophisticated level of cognitive development in which many separate abilities are co-ordinated."
>
> (Gelman, 1972, pp.88-89)

The suggestion is thus that the difference between children at the pre-concrete and concrete operational stages lies in the fact that the younger children do not spontaneously coordinate all the skills and knowledge needed to solve concrete operational tasks such as conservation. This may be because the younger child lacks the capacity to deal with the number of factors involved and their coordination (see Case, 1974a; Halford, 1978; McLaughlin, 1963; Pascual-Leone & Smith, 1969 for an argument re-interpreting the Piagetian theory in terms of the develoment of short-term memory capacity and the consequent number of elements or operations that can be coordinated at any one time). The onset of concrete operationality may thus coincide with the capacity to spontaneously coordinate the logic and other skills necessary for successful handling of tasks such as conservation. By this analysis, if the conditions of a task situation were such as to permit the younger child to coordinate the required factors (assuming that such task conditions exist), then performance characteristic of a more mature stage of cognitive functioning may occur. Such an account accommodates the findings from all three kinds of study discussed, namely those dealing with variations in task feature, with training intervention, and with modification of the language of communication between child and experimenter. Conversely, of course, one might suggest that the older child who has the capacity to spontaneously coordinate the relevant factors may not in fact do so, hence resulting in the appearance of responses characteristic of less mature ways of reasoning (see the discussion on point '3' below).

This perspective is very close to that adopted by the 'skill integrationists' (e.g. Bruner, 1973; Klahr & Wallace, 1973; Schaeffer, Eggleston & Scott, 1974). Each of these studies is based on a model in which the development of various logical capacities is dependent upon the integration of skills or procedures appropriate to specific settings. Integration of these skills permits increasing generalisation to new situations, results in 'chunking' which reduces the demand on available 'memory-space', and allows the development of schemes relating various skill elements. Such a model is still structuralist, in that it presupposes the organisation and coordination of skills in particular ways and the subsumption of earlier coordinations into later ones. However, it differs from the Piagetian perspective in considering that logical capacities such as conservation develop from specific skills rather than as the consequence of a 'spontaneous cognitive reorganisation' such as that associated with the transition between successive Piagetian cognitive structures. What is as yet lacking from this model is an analysis of how these skills become integrated in the particular manner supposed. Such explanation would seem to require either a commitment to associationism, or the positing of more fundamental psychological mechanisms which enable the coordination of skill described. To what extent these mechanisms may differ from the 'general psychological capabilities' described by Piaget must remain a matter for conjecture.

The non-appearance of expected operations. Concern over the validity of the Piagetian stage descriptions has also been expressed as the result of observations that children do not always demonstrate the availability of a given logical operation despite its apparent manifestation on other task situations, or despite the manifestation of other stage-linked patterns of reasoning. This concern has been expressed with regard to both concrete operations (e.g. Lovell, 1968; Lovell & Ogilvie, 1961; Lunzer, 1960) and formal operations (e.g. Lunzer, Harrison & Davey, 1972; Martorano, 1977; Pulos & Linn, 1981; Wason & Johnson-Laird, 1972). Piaget himself (e.g. 1971) noted the 'décalage' or variability in children's performance on different tasks, and introduced the idea of the 'resistance' of some tasks to admit the application of operations which are otherwise available to the child. Where the concrete operational stage is concerned, the appearance of such décalages does not in itself invalidate the Piagetian formulation of integrated structures, as discussed in the preceding section. Since the "development of concrete operations ... can never be dissociated from the intuitive content to which such operations need to be applied" (Inhelder & Piaget, 1958, p.265), then where the content to which the operations under question relate is outside the child's experience or conceptual range, it would not be surprising if the child's ability to solve tasks involving this content should be delayed. Reasoning involving displacement volume, for example, might be delayed relative to the same reasoning with regard to number.

The same argument does not, of course, apply to the formal operational stage, since at this level "the operational form is entirely dissociated from thought content" (Inhelder & Piaget, op.cit.), so that the same logical operations appear "in the most diverse areas".

However, apart from the recognised likely unevenness of appearance of all the formal operational structures during the early phase of the stage, it has been suggested that in any case one must not assume that the formal operational thinker always makes use of formal reasoning. Thus Inhelder states (Tanner & Inhelder, 1960, p.126) that "both adolescent and adult are far from reasoning formally all the time. The attainment of a cognitive stage merely means that the individual becomes capable of behaving in a certain way which was not possible before." The same point seems to underlie Piaget's remarks that:

> "Finally, and above all (for it would be impossible to emphasize this point too strongly), each stage of development is characterised much less by a fixed thought content than by a certain power, a certain potential activity, capable of achieving such and such a result according to the environment in which the child lives."
>
> (Piaget, 1971, p.171-172)

For example, whether or not an individual makes use of formal operational thinking in a given situation may depend on the task's complexity and the individual's familiarity with the task content (e.g. Lunzer, 1965). Peel (1960) suggested that an adolescent or adult who is capable of full propositional (formal) thinking may not use it on any given occasion since such thinking may require 'more effort' than the problem warrants. Consequently the individual may rely on concrete operational thought, or else may 'play a hunch' (Bruner, Goodnow & Austin, 1956). Of course, if the complexity of the task has been misjudged, perhaps because of the individual's lack of familiarity with the subject of the task, such an approach may lead the individual into error, and the researcher into thinking that the individual lacks the appropriate formal strategy.

It is, however, unlikely that perceived 'pay-off' alone determines the non-application of formal operations to a given situation. Lovell (1971) has discussed the question of the characteristics of the content of a task, as opposed to the intellectual structures needed to solve it, which may determine whether or not formal operational thinking is brought to bear on a particular problem. As a result, he reinforces the idea of the possible involvement of such factors as familiarity with the content of a task, credibility or 'direction of belief', and so on (see also Lunzer, 1965). To these have been added other factors such as cognitive style (Case, 1974a, 1975, 1978; Linn, 1978) and expectation or 'mental set' (Linn & Swiney, 1981).

The role of task content. The influence of task content on the child's reasoning at both concrete and formal operational levels has in fact been demonstrated by several workers. Where the concrete operational stage is concerned, such demonstration may not be unexpected in terms of the Piagetian theory, as previously discussed, although too extensive a demonstration of décalage may be less readily reconciled with the model. In the case of formal operational thinking, the demonstration of décalage has typically been regarded as more problematic for the theory (but see the preceding discussion).

That task content can be important even at the formal operational level was shown by Wason and Johnson-Laird (1972; see also Lunzer, Harrison & Davey, 1972) in their demonstration that a task in which the content was familiar was much easier for subjects to solve than was a structurally equivalent problem but which involved unfamiliar or more abstract content. In an investigation of adolescents' understanding of abstract economical and historical concepts and moral dilemmas, Peel (1975, 1978) showed that the ability to deal formally with social problems appeared much later than the ability to handle physical problems in the same manner. From this he concluded that the ability to reason formally was subject-linked and depended upon the possession of appropriate experience and knowledge as well as upon the availability of formal operational logical capabilities. In line with this suggestion, Pulos and Linn (1981) found that groups of 'experts' performed more effectively on 'control of variables' tasks involving content specific to their area of expertise than they did on tasks involving 'unfamiliar' content, even though the nature of the content was irrelevant to the task's solution. The nature of the task seemed, from these studies, to be important in that (a) it may determine whether or not formal thinking is invoked, as suggested also in the preceding section, and (b) it may determine whether a successful outcome can be achieved even if formal operational reasoning is utilised. For example, an individual may be unsuccessful on a formal operational task not because the 'control of variables' schema is missing but rather because the individual lacks the knowledge as to which variables are important in that particular task. Thus Linn and Swiney (1981) found that late adolescents' views of which variables were important in the springs task (a 'control of variables' task designed by Linn and Rice, 1979, cited in Linn, 1982) differed from those presented by the experimenter. Despite being presented with the experimenter's more comprehensive list of potentially important factors, however, the students went on to perform the task as if only their own expected variables were at issue. In another study involving the pendulum task (Inhelder & Piaget, 1958, chptr.4), Pulos and Linn (1978, in Linn, 1982) observed that subjects expected the wrong variables to be important (i.e. weight of pendulum bob rather than length of pendulum), and directed their responses accordingly. From the results of these and similar studies, the authors concluded that whilst knowledge of task content is not part of the Piagetian description of intellectual performance, consideration of this factor is necessary to an understanding of children's actual performance. The same point was made by Wason and Johnson-Laird:

> "From some considerable time we cherished the illusion that ... only the structural characteristics of the problem mattered. Only gradually did we realise first that there was no existing formal calculus which correctly modelled our subjects' inferences, and second that no purely formal calculus would succeed. Content is crucial, and this suggests that any general theory of human reasoning must include an important semantic component."
>
> (Wason & Johnson-Laird, 1972, pp.244-5)

Summary

In summary, it would seem that the evidence against the validity of the Piagetian theory is at best less strong than some critics have suggested, and at worse ambiguous and inconclusive. Nevertheless, the evidence available must provide a caution against adopting the Piagetian theory in toto and as a model which uniquely predicts an individual's intellectual behaviour on a given task (if indeed this was ever Piaget's claim). In particular, it is considered that attention must be paid to the nature and content of the task concerned and to the individual's knowledge and experience with respect to that task. Even this information cannot permit a firm prediction, since much must still depend upon the influence of such factors as motivation to complete the task, perception of the task goal, ease of retrievability of required knowledge, and personality and attentional factors.

At the same time, it is perhaps premature to abandon the

Piagetian position completely. Summarising some of the research findings criticising the Piagetian theory, Entwistle (1979, p.125) remarked that "clearly if children's capacity to exhibit formal operations depends on the wording of the question, the nature of the task, the extent of previous knowledge, and the content of the subject area, we are left with serious problems in describing a child in terms of a 'stage' of intellectual development." Such a comment must be aimed more to those who have followed Piaget rather than to Piaget himself, since it is not clear that Piaget intended that such labelling was either appropriate or desirable. However, the caution is a sound one, but should not be taken to assume that the whole Piagetian formulation is thereby discredited. True, in view of the factors apparently affecting a child's performance on Piagetian and other tasks, the Piagetian stage analysis must not be religiously relied upon. However, it can provide an extremely useful first approximation, and in the absence of any more comprehensive theory (i.e. comprehensive of task and other factors), it provides a viable model which can usefully inform research into the question of children's cognition.

Because of the difficulty concerning the application of the 'stage' notion to individual children, Entwistle took Marton's (1978, in Entwistle, 1979) point that this perspective should be abandoned in favour of its use as a descriptor of how a particular task is approached or what level of understanding is demonstrated in a given situation. This approach has been adopted in particular by Collis and Biggs (Biggs, 1978, 1980; Biggs & Collis, 1982; Collis, 1980) who found such variations in performance across subject areas that they considered it necessary to abandon the idea of stages of cognitive development. "Accordingly, we relabelled the levels with a more descriptive terminology in order to distinguish our levels from Piagetian stages: levels apply to the classification of observed responses .... " (Biggs, 1978, p.3). As a consequence, a taxonomy of 'observed learning outcomes' was developed for classifying responses in a variety of subject areas. Under this model it is thus the response which is classified according to a structure of Piagetian-like levels, and the freedom of the individual to exhibit responses at different levels in different subject areas is thereby allowed.

This 'modification' of the Piagetian model represents an interesting approach. However, while it answers the problem of labelling individuals, it is not clear that it presents a theoretical advance. Without recourse to the Piagetian analysis, the question of what determines the nature of the various levels of response, and by what processes an individual may progress from one level to the next, remains largely unspecified.

Relevance to the Present Research

It was therefore considered that in the present research little would be gained by a total abandonment of the Piagetian model, and indeed that such an abandonment was in any case not justified by the evidence available. However, in view of the various criticisms and the equivocal nature of the research findings both in support of and refuting this criticism, it seemed preferable to keep an open mind on the subject and to allow for the possible modification of parts of the theory and the potential operation of other factors, especially task-related ones. As a result, the theoretical model for children's cognitive functioning adopted for the present research was based upon the Piagetian structuralist theory in so far as the following points were concerned:

(1) The child possesses a kind of mental organisation or 'cognitive structure' which determines the kind of logical thinking available to the child.

(2) This 'cognitive structure' exists at the level of the 'psychological machine', i.e. it is not available to conscious introspective review.

(3) The cognitive structure is not immutable but changes or develops as the result of experience or maturation.

(4) The cognitive structures thus developed are constructed as a consequence of the child's actions on the environment.

(5) A constantly-made error to a given task reflects a way of viewing that task, or handling its solution, which is consonant with the child's current cognitive structure. Analysis of this error provides valuable insight into the child's mode of cognitive functioning.

(6) The development of cognitive structures depends upon the attainment of successively broader and more stable equilibrations brought about by the incorporation or 'assimilation' of new experiences to existing cognitive structures and the consequent modification or 'accommodation' of the latter to these new experiences.

This last point relates to an aspect of the Piagetian theory which has not yet been discussed, namely the functional as opposed to structural properties of intelligence. Not only was Piaget concerned with specifying the nature or structure of successive stages (this aspect having formed the substance of this chapter so far), but he was also concerned to describe the processes by which development took place. This aspect of Piagetian theory has received very little empirical criticism though it should also be noted that little research into this particular question has in fact been attempted (see Brown & Desforges, 1977). The equilibration model, and in particular the role of 'conflict' as an important feature of the equilibration process, has been used by Inhelder, Sinclair and Bovet (1974) as a means of facilitating progress from one stage to the next, and will be considered again in discussing the precedural model adopted for the teaching phase of the present research.

From all these points of view, the Piagetian model was considered a useful one to inform the present investigation. In addition, it was thought useful to bear in mind Piaget's description of concrete and formal operational thinking lest they prove of value in describing or explaining children's functioning with respect to algebra. The description of formal operational thinking in particular has been the source of much debate, to the extent that Furth (1975) has professed himself uncertain of what a 'formal' task is. Different researchers have imposed their own interpretations upon the issue, concentrating variously on such characteristics as levels of abstraction from concrete referents (Lovell, 1974), the difference between abstraction and generalisation (Peel, 1971), and the involvement of second-order as opposed to first-order relations (Lunzer, 1968), 'complex' versus 'simple' inference (Lunzer, 1973), and 'acceptance of lack of closure' together with the ability to handle 'multiple interacting systems' (Lunzer, 1976). Collis (1973c, 1975b), however, has presented an interpretation of concrete and formal operationality specifically in terms of mathematical abilities, and it is to his work in particular that this thesis looks. This description will be discussed in the next chapter.

The Ausubelian Model

Overview

Whilst bearing in mind the Piagetian model of general cognitive capabilities, however, the need was also felt (as expressed earlier) to take into account the nature of the task presented to the child. That the nature of the subject matter itself is important in the question of the child's learning and understanding of a given discipline such as mathematics has already been emphasised by theorists such as Gagné (1968, 1977) and Ausubel (Ausubel, 1967; Ausubel, Novak & Hanesian, 1978). In his earlier work Gagné, for example, stressed the need to specify a structured learning hierarchy of concepts and principles needed to learn a given task, and considered that the nature of this hierarchy is dependent mainly upon the logical structure of the subject itself. Gagné's concern in mathematics, therefore, has been largely with specifying the particular learning hierarchies associated with a given mathematical learning task (e.g. Gagné, Mayor, Garstens & Paradise, 1962) and with defining the 'conditions of learning' relevant to the various steps of those hierarchies, rather than with providing a description of general mathematical abilities as such. Ausubel similarly concerned himself with the importance of the subject matter in determining a child's performance on or learning of a given task, but drew a distinction between objective and subjective knowledge. Thus for Ausubel it is the 'framework of knowledge' that an individual has constructed with regard to a given task that is of crucial importance in determining his or her performance on or learning of that task. It is the "substantive and organisational properties of previously acquired knowledge in a particular subject-matter field that are relevant for the assimilation of another learning task in the same field" (Ausubel et al, 1978, p.29). As a consequence of experience, both incidental and school-taught, each child constructs a framework of knowledge which will direct that child's subsequent learning in that field. Since different individuals may therefore construct different frameworks of knowledge, task-related variations in performance among individuals of comparable age and intellectual capacity may not be surprising, so that Ausubel is thus able to account for the 'decalages' that have proved problematic for the Piagetian theory. By this account, the 'expert' differs from the 'novice' mainly in terms of the knowledge structure which he or she has acquired in the area under consideration, and the primary aim of instruction must be to assist the novice to build up an efficient and effective framework of knowledge in the given domain. The role of theory in this regard is therefore to suggest ways in which such frameworks of knowledge may be most readily constructed, and much of the work of Ausubel and his co-workers has been devoted to that end (e.g. Ausubel 1964; Ausubel & Fitzgerald, 1962; Ausubel, Novak & Hanesian, 1978).

While the role of some factors, such as the need to follow a concrete-to-abstract progression in the construction of each new framework of knowledge, has been fairly readily accepted to be of importance, however, the role other factors such as the posited 'advance organiser' has been disputed (Barnes & Clawson, 1975), and precise guidelines for the construction of knowledge frameworks remain to be specified. Despite this, the notion of a 'framework of knowledge' construct in mathematics learning appears at this stage to be a useful one; by this view, it is the lack of an appropriate operational knowledge structure which underlies children's lack of success in mathematics.

However, while the Ausubelian theory provides some useful constructs and a framework within which to investigate how appropriate knowledge structures may be built up, it is clear that there are other observations concerning child behaviour which the theory may less comfortably subsume. As a result of its emphasis on individually-constructed frameworks of knowledge and its overlooking of any general psychological structure defining general ways of reasoning, for example, the same theory which so cogently accounts for subject-related variations in performance is less effective in explaining consistencies of approach observed in children attempting tasks which make little discipline-specific demand. The operation of some non-subject-linked cognitive structure defining general reasoning patterns seems to be required in order to overcome such objections as this. The results of the CSMS (Mathematics) research also highlight a finding which, while not contradicting the Ausubelian theory, suggests an area in which further investigation is necessary. While a novice in any particular discipline may be expected to be functioning (at least initially) in a manner characteristic of the concrete thinker as Ausubel suggests, the experience of long-term instruction and experience in that discipline would be expected to result in the construction by the learner of more powerful frameworks of knowledge permitting more abstract reasoning within that subject. The finding by the CSMS (Mathematics) team that large numbers of children (approximately 50 percent or more: see Booth, 1981a, p.30; Hart, 1981a) are apparently functioning in mathematics at a relatively low level of understanding despite many years' experience and instruction in that subject, and that little progress in understanding seems to be made over the two to three years of secondary-school mathematics education monitored by the research project, seems to require some additional explanation. To say that for none of these children was the subject 'meaningfully taught' (Ausubel et al, 1978) is in some sense to state the obvious. What is required is an explanation of what constitutes meaningful learning in mathematics, and why it is that current teaching practice in mathematics is apparently so much at variance with the requirements of meaningful learning, if the theory is to be usefully applied to the construction of a model of mathematical understanding.

Relevance to the Present Research

While the Ausubelian model may require extension and/or modification in order to account for observations of the above kind, the notion of a 'framework of knowledge' construct in mathematics learning was considered to be an extremely valuable one which would usefully supplement the (modified) Piagetian model in informing the present study. In addition, various suggestions from the Ausubelian theory concerning the ways in which frameworks of knowledge may be constructed were also felt to be of value: these will be considered in the discussion on the procedural model for instruction which appears later in this chapter.

METHODOLOGICAL PERSPECTIVES

General Considerations

Overview

The influence of both the Piagetian structuralist model and what might be termed the 'constructivist' model of Ausubel (after Kelly's (1955) personal construct theory) is also to be seen in the choice of research methodology. The emphasis on analysing children's errors is based upon the Piagetian view that a consistently made error to a given problem is a marker to the child's way of interpreting and solving that problem, and hence to the kind of cognitive framework within which the child works. 'Functional analysis', or analysis based upon the child's perspective and way of functioning with respect to that task rather than upon the logic of the task, therefore provides insight into the child's cognition, and the clinical interview procedure developed by Piaget was adopted as the best way of achieving this analysis. It was intended, however, that the interviews should aim to ascertain both the child's more general mode of functioning in the area under study, and the particular knowledge of and concepts relating to the specific area of mathematics being investigated.

Having elaborated a hypothesis concerning the child's conceptual and procedural structures with regard to the topic under investigation, ways must be found of both examining this hypothesis and helping the child to construct the cognitive framework necessary to a correct handling of the problems in question. This may be done by such means as the Piagetian training or 'acceleration' study (Inhelder, Sinclair & Bovet, 1974), or the 'teaching experiment' (see Kantowski, 1979; Menchinskaya, 1969a,b). The Piagetian approach is based on the view that:

> "under certain conditions an acceleration of cognitive development would be possible, but that this could only occur if the training procedures in some way resembled the kind of situations in which progress takes place outside an experimental set-up."
>
> (Inhelder, Sinclair & Bovet, 1974, p.24).

Since the Piagetian theory proposes that such development takes place by process of successive equilibrations brought about by the accommodation of 'old' schemes to admit new experiences, the method adopted by Inhelder et al focussed upon the stimulation of this process, particularly by the use of 'conflict'. Experiences are designed by which the use of the inadequate conceptual schemes leads to conflict between contradictory results. The child is thus stimulated to seek resolution of this conflict, by means of a restructuring of the child's conceptual system so that the task in question can be successfully handled.

The teaching experiment also aims to help the child construct the kind of conceptual framework required for the correct solution of the tasks under study. This is done by means of an instructional programme based (as in the Piagetian case) upon the information obtained from interviews concerning the nature of the conceptual system within which the unsuccessful task-performer is operating, and how this differs from that of the successful child. The essential differences between this approach and that of the Piagetian training procedure are that:

(1) emphasis is placed upon the specific content to which the conceptual framework under study relates, and both interviews and instruction are firmly embedded in the subject context;

(2) no prior assumption is made concerning the mechanisms by which conceptual development may proceed - indeed, part of the purpose of the teaching experiment is to make and examine hypotheses concerning the processes by which this learning may take place; and

(3) having developed and tried the instructional programme with individuals or small groups of children, the teaching experiment incorporates the idea of 'large group' verification, in order to confirm the results of individual experiments on a wider quantitative basis.

Application to the Present Research

Since the present research was concerned to admit the importance of specific content and the development of 'frameworks of knowledge' relevant to this content, and since one aim of the study was to develop a teaching module which could be used with whole classes and hence more directly inform curriculum development in the selected area, it was considered that the teaching experiment paradigm was a more useful one to adopt. The adoption of a more subject-biased approach also required that other procedures than the 'conflict' method be invoked to facilitate the construction of the required knowledge frameworks. Suggestions concerning these other procedures could be derived both from Ausubel's work and from the general principles of instructional psychology.

The total approach adopted in the present research therefore comprised two aspects:

(1) the investigation of the child's cognitive functioning with respect to the area under study, using the Piagetian clinical interview method (in the context of the teaching experiment paradigm this phase is referred to as the 'ascertaining' or 'assessment' experiment - see Menchinskaya, 1969b).

(2) The teaching experiment itself, by which an instructional programme based upon the hypotheses derived from the interview phase, and aimed at helping children construct a more appropriate cognitive framework relevant to the tasks at issue, was developed and its effectiveness monitored.

The Ascertaining Experiment

The Piagetian Interview

As indicated, the model used for this phase of the research was the Piagetian clinical interview. The essential feature of this procedure is that it presents a hypothesis-testing situation (Opper, 1977) which allows the interviewer to deduce aspects of a child's reasoning by observing the child's performance on certain tasks and by witnessing the explanation which the child gives. At the start of an interview session, the interviewer has some guiding hypothesis concerning the kinds of reasoning that the child may bring to bear on the problems presented. Various items designed to reveal different facets of the child's thinking with respect to the problem-area investigated are then presented, and the child's responses to these used to test and clarify the experimenter's original hypothesis.

Application to the Present Research

In the present study, the interviews were conducted in two rounds, by which the modified or elaborated hypotheses derived from the first round of interviews were used to guide the second stage of inteviewing. By this means it was hoped to attain a greater degree of specificity with respect to the hypotheses eventually used as the basis for the teaching programme. Also, since the hypotheses of interest concerned the reasons for particular identified errors, the children interviewed were children who were known to be making those errors. While some 'mathematically able' children were also interviewed during the research, this was primarily in order to clarify some particular aspects of the hypotheses formed, and to investigate their involvement in the reasoning processes of 'more able' children.

The standard guidelines for interviewing with regard to establishment of rapport, non-directive probing, etc. (see Opper, 1977; also Collis, 1981) were adhered to. Since part of what constitutes successful mathematics activity requires the child to be conversant with various aspects of mathematical notation and convention, it was necessary to establish the child's interpretation of any symbols used, and the child's facility in mathematically encoding the answers produced. These questions were therefore also built into the interview procedure. Details of the interview schedule used in the present study are given in Chapter 5.

The Teaching Experiment

General Principles

Components of the teaching experiment. As indicated in the earlier discussion, the focus of the teaching experiment is on 'cognitive instruction' (Belmont & Butterfield, 1977), i.e. instruction which aims to lead the child to develop a conceptual framework which will enable the correct solutions of the problems under study, rather than which aims to teach the correct solutions of those problems directly. The questions which guide the researcher are those relating to an identification of the general cognitive or knowledge-based abilities relevant to the tasks of interest which the unsuccessful task-performer lacks, and those relating to the processes by which these abilities may be developed. The assumption is made that successful inculcation of these abilities will result in the successful handling of the tasks under study, and also of tasks of related structure. Since a major function of the teaching experiment is to examine how the chosen process of instruction and the process of constructing the required cognitive framework relate to each other it is necessary to study the changes in mental activity that occur under the influence of instruction, that is to form hypotheses concerning the learning process and hence to monitor the child's interaction with each step of the instructional procedure. From this point of view, the teaching experiment represents an extension of the interview situation. Whereas the latter forms and examines hypotheses about the conceptual framework that the child already has, the teaching experiment forms and examines hypotheses about the processes by which the required new conceptual structures may be attained. The components of the teaching experiment are therefore:

(1) the formation of hypotheses concerning the nature of the cognitive framework required for correct handling of the tasks involved and which the child appears to lack (this being done on the basis of the information obtained from the interviews);

(2) the formation of hypotheses concerning the process by which this required cognitive framework may be developed;

(3) based on these hypotheses, the development of a teaching strategy aimed at stimulating the required learning;

(4) the trial of this teaching strategy with individuals or small groups and the close monitoring of its effectiveness in order to ascertain the psycho-pedagogical reasons for its effectiveness;

(5) subsequent modification and re-trial of the teaching programme; and

(6) verification of the teaching strategy with large groups.

<u>The problem of direct measurement.</u> In order to permit the monitoring of the programme's effectiveness as outlined in (4) above, it is necessary that the researcher observe as directly as possible the child's responses to the instructional process. The teaching experiment must contain prepared probes to test the learning at the

time that it is occurring. The problem of the means by which such direct measurement may be achieved is not a trivial one and has been discussed in detail by Belmont and Butterfield (1977). Among other requirements of direct measurement, Belmont and Butterfield list those of minimising both temporal and logical distance between task and measurement. Temporal distance is minimised when the activity is measured at the time that it occurs; the minimising of logical distance requires that the activity measured be closely related to the conceptual aspect under study, so that it can be considered unlikely that variations in any other factor are accounting for observed variations in the measure. Various techniques suitable for use in laboratory experiments are described, but most of these are not readily applied in the classroom situation. Consequently, Kieran (1981c) has suggested the use of the 'dialogue' approach in the teaching process, as a means of obtaining a more natural probing devise and one which can be used to provide an ongoing analysis of the teaching-learning situation. Since the questions posed can be related directly to the conceptual foundations under study, this technique also fulfils the requirement for logical proximity of measurement.

A procedural model for teaching. As mentioned in the discussion of Ausubel's theory on the construction of frameworks of knowledge, Ausubel had been concerned to consider the processes by which this construction might be most effectively promoted. In this regard, particular mention had been made of the need to follow a concrete-to-abstract progression in the development of each new knowledge framework. Also important in this regard was suggested to be the 'advanced organizer'. Learning, according to Ausubel (Ausubel, Novak & Hanesian, 1978), should be 'meaningful', that is, it must be integrated into already existing cognitive structures. For Ausubel and his colleagues, the "most important single factor influencing learning is what the learner already knows" (Ausubel et al., 1978, frontispiece). The learning task must be related in a 'non-arbitrary substantive fashion' to the learner's existing knowledge structure. In order to do this, the learner must adopt an appropriate learning set. Consequently, Ausubel postulated the importance of the 'advance organizer', an introductory overview which indicates the relationship of the new material in the framework of knowledge already possessed, and which induces a set of higher-level organizing concepts which facilitate assimilation of the new material. The role of the advance organizer has been much debated, and empirical evidence in its support appears to be equivocal (Barnes & Clawson, 1975; Harley & Davies, 1976; but see replies by Ausubel, 1978; Lawton & Wanska, 1977). Part of this debate appears to be founded upon a confusion as to what an advance organizer is. However, it is also possible that some of the discrepancies in reported results may be due in part to a differential effect of the organizer for individuals at different stages of conceptual framework construction. Ausubel (1960, 1978; Ausubel, Novak & Hanesian, 1978) has defined advance organizers as introductory material at a higher level of abstraction, generality and inclusiveness than the material to be learned itself, but which is relatable to presumed ideational content in the learner's current cognitive structure. It may be that different characteristics of the organizer are more crucial for learners at different stages. For example, the child whose conceptual framework is still of a highly concrete nature may benefit most from organizers which are characterised less by degree of abstraction than by degree of generality and inclusiveness. The reverse may be true for 'experts' whose knowledge structure is more formally organized.

Despite the equivocal nature of the evidence in support of the role of the advance organizer, it was felt that the potential value of an advance learning frame of some kind was sufficiently great to merit inclusion in the present programme. In particular, the importance of relating new material to existing cognitive schemes was taken as a sine qua non (see also Skemp, 1968, 1971, 1979). The view of the child as actively constructing cognitive frameworks (cf. Piaget) further requires that the child be actively involved in learning. In addition, the need to incorporate into the design of teaching programmes such principles as feedback and reinforcement, pacing of learning-material presentation, and discrimination training in concept formation, has been variously exhorted by other workers. As a result of such considerations, as well as the established principles of good teaching and the researcher's own experience in this regard, the following guidelines for a procedural model for cognitive instruction were adopted:

(1) Learning must be related to children's existing conceptual structures.

(2) Children learn better given an appropriate 'learning frame' (such as an advance organizer).

(3) Children's learning must be 'active'. This does not mean that the child must necessarily physically manipulate material (Anthony, 1977; Inhelder, Sinclair & Bovet, 1974), nor that activity per se is what is critical, but rather that the child must be <u>actively involved</u> in the learning process.

(4) The new material must be presented in small sections so as to keep down the number of items of information to be attended to at any one time, and so minimise 'working

memory load' (see Case, 1974a, 1975, 1978 for a discussion of the relationship between conceptual develoment and information processing capacity or 'memory space').

(5) Practice is necessary to allow 'chunking' (cf. Miller, 1956) of information and conceptual elements and to permit a strengthening of the links being established between new material and existing conceptual structures.

(6) Feedback and reinforcement is likewise needed to both form and strengthen these links.

(7) Children learn concepts better by appraising both instances and non-instances of the given concept.

Application to the Present Research

The components of the teaching experiment as outlined in the previous section were adopted in full, with the initial teaching trials being conducted with small groups rather than individuals in order to permit information from a larger number of children to be obtained. This phase was followed by modification of the teaching programme in the light of this information, and the programme's subsequent re-trial and verification with large groups.

The requirements for direct measurement were met, as far as possible, by adopting the 'dialogue' approach as, for example, discussed by Kieran (1981c; see also Herscovics & Kieran, 1980), and also by organising the teaching around a serie of worksheets and answer-recording sheets so that a full written record of each child's responses to different sections of the work could be obtained.

The guidelines for a procedural model for 'cognitive instruction' outlined above were likewise adhered to. The precise way in which these principles were translated into classroom practice is described in the notes to the teaching module in Appendix 5. Details of other aspects of the methodology followed are given in Chapters 5 to 8.

CHAPTER 4: EMPIRICAL PERSPECTIVES - CHILDREN'S UNDERSTANDING OF ALGEBRA

The Nature of Algebra

Despite the fact that algebra appears, to greater or lesser extent, in all secondary school mathematics syllabuses, there is sometimes some difference of opinion among those involved in the learning and teaching of algebra as to what algebra is, and what are the aims in teaching it. In 1956 Saad (in Saad & Storer, 1960) undertook a survey of children's understanding and attainment in algebra and noted that children tended to view algebra as a set of manipulative techniques, a view which also seems to be currently held by teachers (Department of Education and Science Report, 1980). Gattegno (1980, quoted in Wheeler, 1981), on the other hand, saw algebra as the essence of all mathematical activity, a subject whose content matter was "operations on operations ... the dynamics of relationships" (p.29). In similar vein, Wagner (1981b) suggested that algebra, like all higher mathematics, is "the study of relations, from simple equations and functions to complex patterns and structures" (p.107). Noting the extension of the term 'algebra' to cover work on such aspects as sets, groups and matrices which as apparent in many new school curricula, Bell and O'Brien (1981) nevertheless drew attention to 'an important objective' of teachers of algebra in the early secondary school, namely the development of an understanding of algebra as 'generalised arithmetic'. By this view, algebra is to be regarded, at least in its elementary stage, as the representation in general form of the operations and structures of arithmetic, as "the expression in algebraic form of quantitative relationships, either given in verbal form or induced from arithmetic examples" (South Notts Project, 1980, p.5), as well as the manipulation of the expression so constructed in order to derive new aspects of the relationships which have been symbolised in this way. It is this aspect of algebra which was taken as the subject of investigation by both the CSMS research and the present study.

Prior to the early 1970's, when the CSMS work in algebra commenced, there had been little research in algebra which specifically looked beyond the acquisition of algebraic skills to a consideration of the conceptual framework within which those skills might be constructed. Curriculum development in algebra proceeded upon the basis of content analysis, and the intuitions of good teachers, and while this led to some major innovations in the teaching of this topic, it seemed that many children still had difficulty in understanding algebra, a difficulty which was informally noted by teachers and which has since received more formal recognition (e.g. Assessment of Performance Unit, 1980, 1981, 1982; Committee of Inquiry into the Teaching of Mathematics, 1982).

Evidently, consideration of the content matter and logical ordering of algebra is in itself insufficient where the question of algebraic instruction is concerned. In making decisions about the way in which algebra tuition should proceed, therefore, it seemed necessary to look beyond the subject matter itself to the characteristics of the learner. As a result, the interest of mathematics educators and researchers began to centre upon the conceptual viewpoint of the learner. The issues which were consequently focussed upon were those relating to the nature of the cognitive structures which children bring to the study of algebra and how these affect what is subsequently learned, the meanings that children ascribe to algebraic concepts and symbols, and the kinds of strategy or procedure that are used by children in solving algebraic problems. Only by gaining an insight into the kinds of conceptual framework which the learner brings to bear in attempting to learn this topic was it thought possible to begin to suggest ways in which it might be more meaningfully taught.

The Development of the CSMS Algebra Test

It was within this context that the CSMS investigation into children's understanding of generalised arithmetic commenced. As described in Chapter 2, the CSMS team approached the question of the relative levels of difficulty of various notions in elementary algebra by developing a test of understanding in algebra which could be administered to a large sample of children. By this means it was hoped to describe both the kinds of concepts that children acquire and the order in which these are typically attained. The approach was Piagetian in its assumption that children "went through the same stages in their grasp of a given concept and that their level of understanding was consistent across different concepts" (Küchemann, 1980 p.13), and also in its focus on children's thinking as shown by their responses and explanations.

The Interpretation of Letters

The work of Collis. The decision to limit the CSMS algebra test to notions of generalised arithmetic was taken in the test's development (Küchemann, 1980, pp.16-17). In devising the test, Küchemann was influenced in particular by the work of Collis (1969, 1971, 1973c, 1975a,b), who had argued that one of the difficulties which children have in algebra relates to the abstract nature of the elements used. Collis had in fact set out to apply the Piagetian analysis of concrete and formal operational stages (Inhelder & Piaget, 1958) to the particular context of mathematics. The abilities which Collis concentrated on initially as differentiating between concrete and formal operational thinkers were those concerned with the degree of reliance on 'reality'. Concrete operational thinking, Collis reminded us, is restricted to concrete-empirical experience, one consequence of which is that the child considers only what is empirically verifiable. Consequently, where mathematical notions are concerned, Collis suggested that concrete and formal operational thinkers would differ in their ability to handle abstract elements and operations (Collis, 1969, 1971), so that a progression would be discerned in the child's ability to handle small numbers of immediate experience (such as 3 or 7), larger numbers which lie outside the immediately verifiable range (for example, 758), and finally algebraic elements. These latter elements would in turn be handled initially as standing for particular unique numbers and only later as 'generalised number' and eventually as 'variables' (Collis, 1975c). Interacting with this development in abstraction with regard to the elements of mathematics, Collis described a similar progression in terms of degree of abstraction of operation. While for the early concrete operational thinker each operation is viewed as having a distinct physical meaning such as 'taking away' or 'sharing', the older child begins to work with operations as such, until the final stage at which an operation may be purely symbolic with no necessary relationship to physical reality (Collis, 1973c).

Closely related to this progression was, Collis suggested, a growth of tolerance for, or acceptance of, lack of closure (Collis, 1969, 1973c, 1975a). Part of what is meant by the physical interpretation of an operation is the requirement for the operation to be actually performed and the result observed. Hence at the early concrete operational level, the operation '2+3' has meaning only when the action is performed and the replacement '5' is made; arithmetic closure is thus necessary to the child's understanding of the operation. At the late concrete operational level, closure is also required, but the child can now handle a series of operations such as 2+9+6, provided that each operation is closed in turn before attempting the next operation (i.e. 2+9+6 = 11+6 = 17). Since the child now begins to regard the outcome of an operation as necessarily unique, the meaning of operations upon numbers outside the empirically justifiable range (such as 897+538) can now be understood.

Halford (1970,1978) has interpreted this in terms of the number of judgements that can be combined at one time. He suggested that concrete operational thinkers can only combine numbers or judgements two at a time, hence the requirement for closure on a single operation between two elements which Collis has demonstrated to hold at this level.

By contrast, the formal operational thinker's ability to think of operations in a 'formal' (not-necessarily-related-to-reality) sense enables the child at this level to separate the operation from the elements upon which it is performed. The operation thus acquires meaning as a unit, apart from the elements to which it relates. At this stage the child is able to refrain from closure, and considers the relationships involved in operational statements, rather than merely contemplating the obtaining of a unique result. Hence the generalisation a/b = (nxa)/(nxb) is seen to be true in terms of the relationships which obtain between the operations involved, and is therefore independent of the nature of the elements (be they integers, fractions, irrational numbers, and so on), rather than being a generalisation which is true because of a property of the specific elements involved. The child can also now work with formulae of the kind $v = lxbxh$ (where v represents the volume of a cuboid and l,b and h its length, breadth and height respectively), which can be used not only to obtain specific results by substitution, but also to consider the effects of various combinations of transformations on the formula, such as simultaneously increasing l and b and decreasing h. This stage of operating is preceded by a 'concrete generalisation' stage (Collis, 1973c, 1975a), in which the child can also work with such formulae, but only in order to obtain particular answers by given substitutions. In this case the child handles these apparent formal statements not by formal reasoning as such, but rather on the basis of argument by analogy from concrete examples. While the statement represents a formalization of the volume-finding rule for a cuboid, the procedure which underlies its use is essentially concrete.

A criticism of Collis' interpretation. The development of the child's reasoning ability in algebra was thus related in this way to the child's 'degree of reliance on reality', as indeed was the child's progression in handling small numbers (defined by Collis as numbers less than 10) prior to large numbers. Whilst neither the value of such an explanation,nor the empirical fact of such a progression in understanding is disputed, it may be that a slightly different view of its basis is more accurate. For example, there are certain problems with the interpretation given by Collis of the child's successively acquired facilities in handling small as opposed to large numbers. Thus the term 'concrete' in the Piagetian description does not imply that the child must have objects to manipulate, nor that an immediate physical representation is necessary in order to ascribe meanings to symbols. Rather it implies that the child's thinking is restricted to things and events that can be acknowledged to be 'real'. From this point of view, concrete-as-opposed-to-formal is not the same as concrete-as-opposed-to-abstract, and there is nothing inherently 'less real' about larger numbers, compared with small ones, or indeed even about numbers represented by some other symbol such as letters. What does differ where these various kinds of data are concerned is the degree of complexity of the symbols used, the manner in which meaning is ascribed to them, and the way in which they are used. The symbol '3' may be directly matched with a particular set of objects or perceptual configuration, and this is the manner in which meaning is attached to the symbol. The same matching procedure is not economical in the case of '379', and its meaning is therefore assigned in a different way. Since '379' is a more complex symbol than '3', involving as it does notions of place-value as well as the values of individual numerals, it would not be surprising that the former presents more difficulty than the latter. However, both symbols represent particular unique 'real' values, and there is nothing in Piaget's theory that one kind of data should be less accessible to understanding than the other.

The problem concerning the use of algebraic elements as the data of mathematical operation is suggested to be similarly complex. Hence, while the 'meaning' of a letter can be considered apart from the use to which it is put, the two aspects are closely interrelated, and part of the difficulty which children appear to have in handling letters may relate not so much to the 'abstractness' of the letters per se, but to the fact that letters are often used to describe a pattern in data. Thus the statement '2x + x = 3x' describes the pattern which underlies all statements of the form '2x4 + 4 = 3 x 4';

appreciating the 'generalised number' aspect of letters is therefore difficult to separate out from the recognition and statement of the pattern which the letters are used to describe, and indeed from the perceived need to do so. The difficulty observed with respect to the meaning of letters issue may therefore relate as much to difficulties associated with the recognition and statement of data patterns as to the 'abstractness' of the letter symbols per se.

An alternative account. This interrelationship between the meaning ascribed to letters and the way in which letters are used, and the corresponding ability of children to handle letters and algebraic statements at a generalised number level, may, however, still be accounted for in Piagetian terms, even if at the expense of modifying Collis' interpretation. Such an account requires focussing upon the direction of thought between reality and possibility rather than upon the child's degree of reliance on reality per se as Collis suggested. The critical property of formal operational thought is in fact not so much that it extends to abstractions but that it represents a reversal of the direction of thinking between reality and possibility in the subject's method of approach (Inhelder & Piaget, 1958; Flavell, 1963). For the concrete-operation child, there is also a sense in which possibility can be dealt with, but only as a 'simple potential prolongation' of actions applied in a given situation, such as imagining the 'possibility' of continuing a seriation having begun it. Thus concrete thought deals with 'possibility-as-an-extension-of-the-actual-situation' (Inhelder & Piaget, 1958, p258), and the child proceeds to the 'possibility' (in this restricted sense) from the reality. By contrast, for the formal operational thinker, it is reality which is secondary to possibility; 'reality' is thus that particular event in a whole set of possible events, which has actually occurred, and thinking thus progresses from what is (logically) possible to what is (empirically) 'real'. It is this reversal of the direction between reality and possibility which Piaget regards as the most general characteristic of formal thought (ibid, p255), underlying as it does most of the abilities demonstrated by the formal operational thinker. This difference in approach may be seen to have implications for the way in which concrete and formal operational thinkers view both 'variables' and generalised statements. For example, for the concrete-operational child, it might be predicted that a symbol representing a variable would be interpreted as having that particular value appropriate to a specific ('real') case; as a consequence of the ability to deal with 'possibility-as-an-extension-of-the-actual-situation', however, the child would also be able to conceive of that symbol possessing another, different value in another similar case. The child's notion of algebraic variable, therefore, would be of a set of particular individual values, each appropriate to a set of particular individual 'real' cases. In similar fashion, it might be expected that the child's concept of a 'rule' or generalised statement would be that of a 'pattern' statement which describes what would be true in <u>the next comparable 'real' situation,</u> rather than describing what is true of all possible situations of that kind. Such a pattern-statement would be derived directly from (and only from?) a consideration of a set of particular-value cases. This is, of course, the way in which general statements are typically developed in mathematics classes: the child investigates several particular numerical example of a given rule, records the numerical values involved, inspects the values in order to discern the relationship involved, and then symbolises this relationship as a rule or generalised statement. The only problem with such an approach would seem to be that, from the concrete-operational child's point of view, it makes nonsense of the need for variables or for generalised statements. If all the rule does is express what would be true in the next (real) instance of that class of events, then it would seem to constitute an unnecessary complication. All one needs to do is determine what the particular numerical parameters of the next instance are, and then solve the problem accordingly.

By contrast, in progressing in thought from what is logically possible to what is empirically real, the formal-operational thinker is suggested to be able to conceive of a variable as assuming a range of values, only one of which might be applicable to any particular given situation. Similarly, a generalised statement is perceived by such a thinker as a 'summary' statement of the relationship appropriate to a given class of events; any particular problem situation is merely one instance of the range of possible occurrences of that class. Again, for the formal thinker, reality is subordinate to possibility. The formal operational child, therefore, unlike the child at the concrete operational stage, begins with hypotheses, whose truth or falsehood is recognised to be irrelevant to their status as hypotheses, but which is recognised to be open to empirical determination if required. This change in approach underlies the development of hypothetico-deductive reasoning, so that "instead of deriving a rudimentary type of theory from the empirical data as is done in concrete inferences, formal thought begins with a theoretical synthesis implying that certain relations are necessary, and thus proceeds in the opposite direction" (Inhelder & Piaget, 1958, p251). In the case of algebra, the direction of thinking is <u>from</u> the generalised state <u>to</u> the consideration of particular numerical instances which may reflect upon its truth or falsity.

<u>Application to the CSMS research</u>. However, despite the possible variation in detail of the explanation underlying the child's progression in understanding algebra, the main point derived from this analysis is that development of understanding in algebra may correspond to a progression in the ways in which letters are interpreted and operated upon, and in the ways in which relationships expressed in algebraic terms are handled. Consequently, the CSMS algebra test was developed primarily on the basis of the meanings that children might give to letters. Attention was therefore focussed on the notion that difficulties in algebra may largely "stem from the extent to which the elements lack meaning for the child", and that children "may give different meanings to the letters, which in turn would affect item difficulty in that some items might be solved in unexpected ways" (Küchemann, 1981, p.103). As the result of the preliminary CSMS testing (based on Collis' analysis of level of letter interpretation) and interviews with individual children held as part of the test development programme, Küchemann was in fact able to discern six categories of letter usage:

Letter evaluated:	This category applies to a response where the letter is assigned a numerical value from the outset.
Letter not used:	Here the child ignores the letter, or at best acknowledges its existence but without giving it a meaning.
Letter as object:	The letter is regarded as a shorthand for an object or as an object in its own right.

Letter as specific unknown:

The child regards a letter as a specific but unknown number, and can operate upon it directly.

Letter as generalised number:

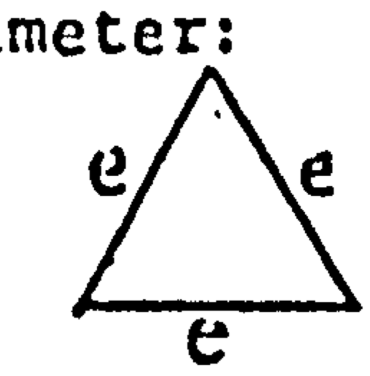
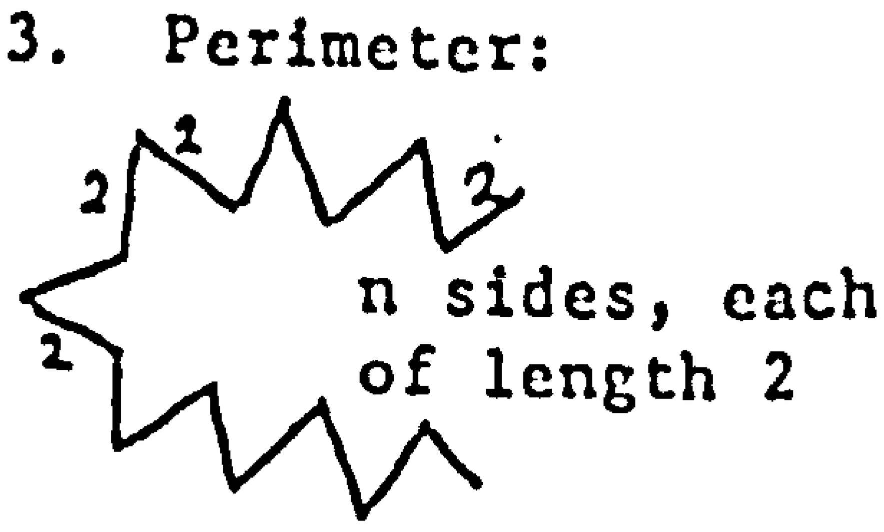
The letter is seen as being able to take several values rather than just one.

Letter as variable: The letter is seen as representing a range of unspecified values, and a systematic relationship is seen to exist between two such sets of values.

These categories were used both to construct the CSMS algebra test and to describe the levels of understanding revealed by the test analysis. For example, the five CSMS algebra items shown in Table 4.1 were suggested to represent different categories of letter usage, as described above, in terms of the minimum level of letter interpretation required for their solution. Thus item 1 in Table 4.1 requires only that the child evaluate the letter directly. Item 2 can be solved by viewing the letter as an object, which merely has to be 'collected up', a way of handling letters which is not, for example, appropriate to item 3. In the latter case it is the '2's' which must be collected up, and the letter must here be understood to represent a value, albeit perhaps a specific value, which determines the number of 2's which must be so collected. While the <u>form</u> of the answers to items 2 and 3 is identical (namely '3e' and '2n' respectively), it can therefore be seen that the answers can nevertheless be differentiated in terms of the minimum level of interpretation which must be afforded the letters involved. In a similar way, item 4 requires not only that the child recognise that the letter can assume a numerical value,

Table 4.1

Examples (Abridged) of CSMS Algebra Items Illustrating Different Categories of Letter Usage

Item	Item
1. $a + 5 = 8$ $a = ?$	2. Perimeter: 
3. Perimeter: 	4. What can you say about c if $c + d = 10$ and c is less than d 5. Which is the larger, 2n or $n + 2$? Explain ....

but also that it can represent a whole range of values, a requirement which was not necessary in the case of item 3. Item 5 goes further than item 4 in its turn, by requiring that the child take into account the <u>relationship</u> involved, in appreciating that the relationship between the two expressions changes as does the value of n. In this respect the item represents a 'letter as variable' usage rather than just a 'generalised number' usage where (as in item 4) the letter is seen as taking on several value instead of one alone. By varying the item demand in this way, it was hoped to examine children's understanding of the meaning of letters in algebra.

The Role of Structural Complexity

<u>The influence of the neo Piagetians.</u> At the same time, attention was also given to the question of the 'structural complexity' of tasks (see Collis, 1975c; Halford, 1978), in terms of the number of elements or operations that had to be dealt with in handling that task. McLaughlin (1963) had suggested that children at different Piagetian stages also differed with respect to their short-term memory capacity, and hence with regard to the number of elements, concepts or operations that could be considered at any one time. He further suggested that an analysis of the various Piagetian stage-related tasks revealed a progression in structural complexity in this respect which precisely matched this development in short-term memory capacity. Consequently it was proposed that the development in cognition described by Piaget could be explained in terms of the growth in the child's memory capacity. This approach, but focussing upon 'processing' capacity rather than 'storage' capacity was also adopted by Pascual-Leone (Pascual-Leone & Smith, 1969; see also Case, 1972a,b, 1974a; Halford, 1978). Halford in fact attempted to apply

this model to the definition of cognitive development stages in school mathematics, thereby subsuming 'the various characteristics of concrete and formal reasoning under the single factor of operational complexity' (Halford, 1978, p.298) and ignoring the question of the nature of the elements involved. In an experiment in which mathematical tasks of varying degrees of structural complexity were given to different age groups of children from approximately 10 to 17 years of age, Halford found evidence to support the view that increasing structural complexity resulted in decreasing facility, at least for the younger children, and that this decrease was of the kind predicted by the informational processing expectations of the model. Consequently, fewer children in the younger age group were able to successfully handle items such as item 2 in Table 4.2, which contained two operations, than were able to deal with the less complex item 1 which involved only one operation. That the two items had comparable facility levels for the group of 17 year olds was explained by the fact that the older children were at a stage of development at which their processing capacity had expanded to the size necessary for a satisfactory handling of the more complex item. As a result, these children were just as able to handle the two-operation item as the single-operation one, so that no resulting difference in facility was observed. From this point of view, it seemed possible that 'threshold' effects with respect to structural complexity might be obtained, whereby children at a given level of cognitive maturity may be able to handle tasks up to a given degree of complexity, but not beyond.

<u>Application to the CSMS research</u>. In developing the CSMS algebra test, therefore, it was thought necessary to consider the question of structural complexity of item as well as the level of letter

Table 4.2

Examples of Items from Halford's (1978) Study
Illustrating Variation in Structural Complexity

Item	% Facility	
(Find the operation)	10 years old	17 years old
1. 3o6 = 9	84	97
2. (4 * 2)o3 = 2	51	91

interpretation required for the item's solution. Variation in structural complexity was related mainly to the number of elements or operations contained in an item, as for example in the two items illustrated in Table 4.3, where 'structural complexity' is defined in terms of the number of (different) letters involved. Item 2 in Table 4.3 thus represents an item of greater structural complexity than item 1.

Results of the CSMS Algebra Research

A test paper of 50 items (counting each part of every question separately), of the kind illustrated above, and designed to examine the ways in which children interpreted letters and their handling of items of different structural complexity, was consequently constructed. This test was then administered to a large sample of approximately three thousand children aged from 13 to 15 years in the second to fourth years of secondary schooling (see Küchemann, 1980 for details of sampling and test administration).

Levels of Understanding

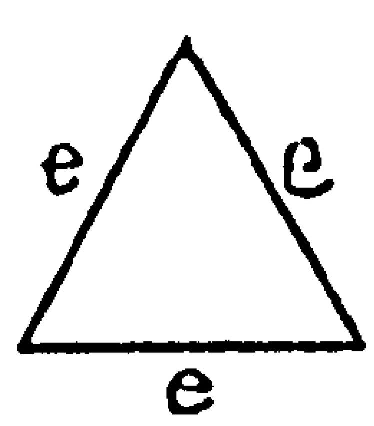
As a result of this testing, Küchemann was able to define a hierarchy of four 'levels of understanding' in which these levels were described in terms of an interaction between complexity and category of letter usage (Küchemann, 1980, 1981; see also Appendix 1B for a description of these levels). Of the six categories of letter usage upon which the test was based, the first three (letter evaluated, not used, or treated as object) in fact describe ways of avoiding generalised arithmetic, by not using the letters as representing unknown numerical values. Küchemann suggested that in order to be able to handle algebra in any satisfactory manner, the child must at least be able to view letters as specific unknowns (Küchemann, 1978).

Table 4.3

Examples (Abridged) from CSMS Algebra Test Illustrating Variation in Structural Complexity

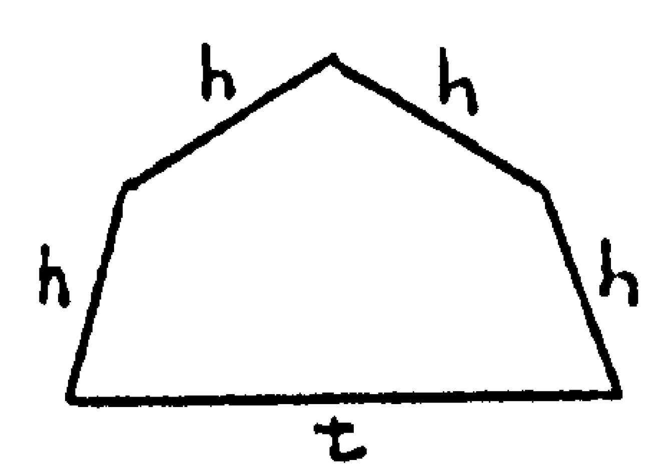
Item 1

Perimeter:



Item 2

Perimeter:



He also drew attention to the fact that while these different ways of handling letters could be viewed as partially ordered, they did not form a strict hierarchy (Küchemann, 1980). Thus the interpretation made of a letter may be expected to depend upon the context in which the letter appears, and the same child may view letters differently in different problems.

Nevertheless, it was apparent that items at the lower two levels of understanding defined by CSMS could largely be handled by the child on the basis of a 'weaker' category of letter usage than letter as specific unknown. It was only at the third level of understanding that the view of letters as necessarily representing numerical values, albeit specific ones, was essential to the correct solution of the items. An analysis of the percentages of children tested who appeared to be operating at each of the four levels described (see Table 4.4) showed that a significant number of children appeared not to be able to 'handle algebra in any satisfactory manner' if this was accepted to require the ability to view letters at least as specific unknowns. Furthermore, while there was a notable improvement between second and third year children in this regard (as shown by the decrease in percentage of children operating at the lower two levels, see Table 4.4), there was still a relatively large number of children who even by the fourth year of secondary schooling had not progressed beyond the lower one or two levels. Indeed, it appeared that the level of understanding as defined by the CSMS algebra hierarchy in many cases improved very little as the child progressed from second year (aged 13) to fourth year (age 15), an observation that was supported by the results of a longitudinal study also carried out as part of the CSMS research (see Table 4.5).

Table 4.4.

Levels Attained by CSMS Sample : Algebra

Year Group	Level[a] 0	1	2	3	4
2nd Year (Age 13)	10	50	23	15	2 per cent
3rd Year (Age 14)	6	35	24	29	6
4th Year[b] (Age 15)	5	30	23	31	9

a. Level 0 is the lowest and Level 4 the highest level-descriptions of the levels are given in Appendix 1B.

b. Percentages do not total 100 due to rounding error.

Table 4.5

Progress of Individual Children in Algebra
From Second Year (Age 13) to Fourth Year (Age 15):
Results from CSMS Longitudinal Study

Number of Levels Moved	Number of Children (N = 105)	%
0 levels	42	40
1 level	51	48
2 levels	9	9
Regressed[a]	3	3

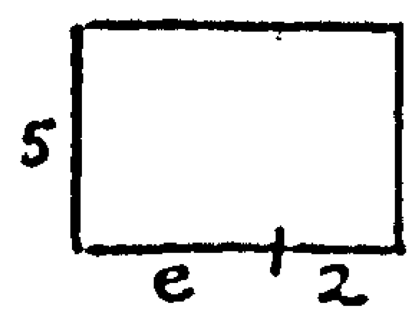
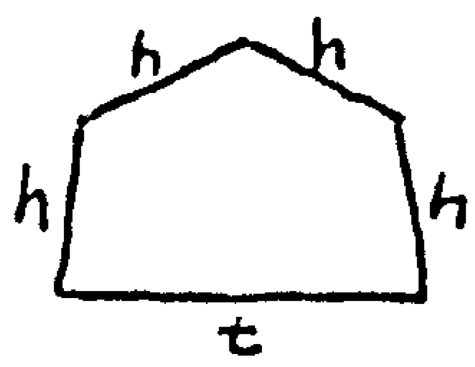
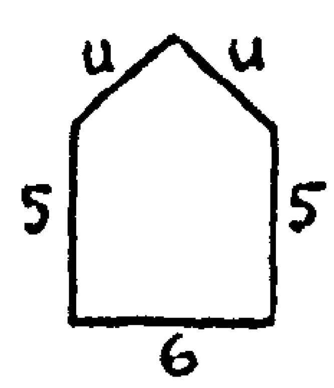
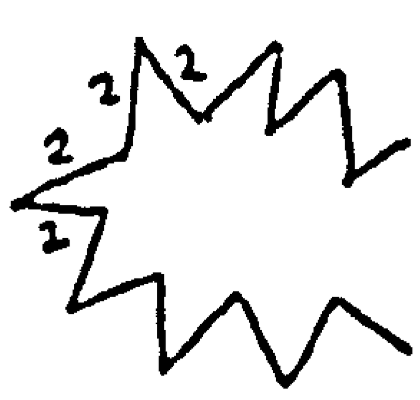
a. A regression is a failure in 1978 to achieve the pass mark of 2/3 on a group of items which had been passed in 1976.

Common Errors in Elementary Algebra

Besides determining the number of children in each age group who had correctly solved each item, the CSMS teach also coded various incorrect responses and ascertained the incidence of each of these. As a result, it was possible to identify certain errors in generalised arithmetic which were made by large numbers of children, in some cases by 40 percent of more of the children tested. Examples of these high-frequency errors in algebra are given in Table 4.6. In many cases the error produced could be related to an inappropriate category of letter usage in handling that item. For example, giving the answer 7ab or 8ab to the item 2a+5b (item 7 in Table 4.6) may reflect the child's treating the letters as objects which may simply be collected up. Similarly, providing the answer 7 to the item 'Add 4 to 3n' (item 5 in Table 4.6) may be regarded as an instance of not using the letter, but rather ignoring its presence and operating simply upon the numbers involved. In the item requiring the perimeter of an n-sided figure in which all the sides are 2 units long (item 4), a sizeable proportion (25 percent) of 13 year olds gave straight numerical answers (32, etc.) which seemed to have been obtained by substituting the actual number of sides in the diagram for 'n', or, less frequently, gave the answer '28', which appeared to derive from a substitution of 14 for n, representing the ordinal position of n in the alphabet. Both of these ways of handling the letter may be regarded as 'letter evaluated' techniques. Viewing letters as specific unknowns rather than generalised number or variables may account for the answer 'never' to the question of whether L+M+N can ever equal L+P+N (item 8, Table 4.6), and may also account for giving a single valued answer rather than a range of values to item 9 (c+d=10 and c is less than d. What can you say about c?). Consequently,

Table 4.6

CSMS High Incidence Errors Selected for Study by SESM (Algebra)

	CSMS Item (Abridged)	'Error' Answer[a]	% Giving Error Answer[b]	Facility[b]
1.	Area of: 	5e2, e10, 10e, e+10	42	7
2.	Perimeter: 	hhhht, 4ht, 5ht	27	57
3.	Perimeter: 	uu556, 2u16	20	28
4.	Perimeter: (n sides of length 2) 	32 to 42	25	9
5.	Add 4 onto 3n	3n4, 7n 7, 12	45 17	22
6.	Multiply by 4: n + 5	4n5, n45 n+20, n+9 20, 9	12 39 16	8
7.	Simplify if you can: 2a + 5b	7ab, 8ab	45	29
8.	L + M + N = L + P + N True: always/never/sometimes	Never	55	11
9.	c + d = 10 and c is less than d. c = ?	single value 0, 1, 2,3,4 (or 5)	43	7

a. Only the main errors for each item are shown. In some cases these categories comprise errors which were coded separately by CSMS.

b. Data given are for the sample of 13 year olds.

children's incorrect responses could be used in the same manner as their correct responses in order to throw light on their possible views of the meanings of letters in algebra.

Contributions from Other Research

The Interpretation of Letters. That the interpretation of letters was not a straightforward matter for many children had also been indicated by other researchers. Wagner (1977, 1981b), for example, found that even 17 year old students may consider that the value represented by a letter is in some way associated with the actual letter used. In a study in which twenty-nine students aged from 12 to 17 years were asked if different solutions would be obtained from pairs of equations such as:

$$7w + 22 = 109$$
and
$$7n + 22 = 109$$

only 50 percent of the 12 year olds and 80 percent of the 17 year olds appeared to regard the solution as being necessarily the same. The remaining students, described as equation 'non-conservers' by Wagner, answered either that the solution of the first is greater than that of the second, because 'w' comes later than 'n' in the alphabet, or that it was impossible to tell until both equations had been solved. These results indicated that some children at least did not realise that the value of an unknown is independent of the letter used, but rather appeared to believe that 'changing a variable symbol implies changing the referent' (Wagner, 1981b, p.116), and also that there may be a correspondence between the linear ordering of alphabet and number system. These results need to be treated with some caution: Wagner herself (pp.115-6) drew attention to the small size and somewhat arbitrary nature of the sample of students used in the study, and also discussed the possible effects of the way in which the conservation questions were phrased (the wording used was deliberately misleading - 'which is larger, w or n?'). Donaldson (1978) has drawn particular attention to the way in which a child's response can vary according to the way in which a question is put, pointing out that children do not approach experimental situations in a totally objective manner, but rather strive to make 'human sense' of the situation, taking into account such variables as perceived experimenter intent. Certainly, when Kieran (1981b) gave the same problem to ten 12 year old pupils but changed the question to 'Are the solutions to these two equations the same or different?', all ten indicated that the solutions were necessarily the same. However, Kieron's sample was also small, and unlike Wagner's subjects, had had no previous exposure to algebra, so that the two groups differed in experience as well as in precise nature of the task set. Despite these somewhat equivocal findings, however, the possibility must nevertheless be recognised that children may endow letters in algebra with connotations other than those intended by the text book or teacher.

Letters have, in fact, been used in algebra by mathematicians in somewhat different ways (see Wagner, 1979, 1981a). Harper (1978, 1979, 1980, 1981) touches upon this point in his analysis of the way in which letters have historically been used in mathematics. Harper distinguished in particular two categories of letter usage which correspond roughly to Küchemann's 'specific unknown' and 'generalised number' or 'variable'. The first of these he labelled 'Diophantine', after Diophantus (circa 250 AD), whose use of letters was essentially as classical (i.e. specific) unknowns, and the second was termed 'Vietan', after Vieta (circa 1600 AD), to represent the notion of a letter as a 'given', or value which does not need to be specified.

Elsewhere this latter usage is described as a 'non-ordered numerical entity' (Harper 1980), akin to Vieta's 'species' by which it was hoped to impart the notion of an entity which existed in its own right and was not a specific, albeit as yet undetermined, number (in this respect the non-ordered number is akin to Küchemann's 'generalised number'). Hence using a letter to represent a specific unique (though currently unknown) value represents a Diophantine use, as in the case of the student who, given the following problem:

> "If you are given the sum and difference of any two numbers, show that you can always find out what the numbers are. Make your answer as general as you can." (Harper, 1981, p172).

solves it by choosing a particular numerical sum and difference, writes down two equations containing two unknowns (e.g. $x + y = 10$, $x - y = 4$) and solves them for the particular numerical values of x and y. By contrast, the student who produces a solution by first writing the general equations:

$$x + y = a, \qquad x - y = b$$

and then 'solving' these to yield the general solution:

$$x = \frac{a + b}{2}, \qquad y = \frac{a - b}{2}$$

reveals a Vietan useage of letters. Harper (1981) suggests that 'algebra' be considered to begin when the letter is first used in this latter manner, considering the Diophantine usage to be more akin to 'arithmetic'. By testing grammar school pupils ages from 11 to 16 years, Harper (1978) showed that there was considerable variation at each age group in the degree of sophistication of letter usage as thus defined, and indicated that a relationship could be discerned between level of letter usage and the successful solution of certain algebraic problems.

Structural Complexity

As anticipated, the factor of structural complexity had featured in the definition of children's levels of understanding described by CSMS. However, evidence from other studies suggests that this factor is itself somewhat complex. Halford's (1978) work on the effects of varying structural complexity has already been referred to. While he was able to demonstrate age-related differences in the ability to handle items of increasing complexity, it was also clear that structural complexity was not the only factor involved in producing those effects. For example, the two items shown in Table 4.7 had the same degree of complexity and were both classified by Halford as 'formal'. However, there was a marked difference in facility between the two items and this was observed at both age-levels studied. Although not discussed by Halford, it was clear that some factor other than structural complexity was involved. In this case it could have been that the difference was due to the difference in nature of the elements involved, as suggested by the Collis-Küchemann analysis.

Other research, however, suggested that there may be yet another aspect to the situation. Firth (1975), for example, used items such as those shown in Table 4.8 in order to test children's knowledge of equation manipulation. While the logical structure of each item is the same, and each item involves two letters, it can be seen from a consideration of the facilities that the items are certainly not the same as far as the pupils are concerned. Similarly, the three items from Ekenstam and Nilsson's (1979) study shown in Table 4.9, in which the children were asked to solve the equations, indicate that there are psychological factors affecting the understanding of such items which go beyond that of mere structural complexity, and which do not relate specifically to the inclusion of algebraic terms per se. In

Table 4.7

Examples of Items of Equal Structural Complexity from Halford's (1978) Study

Item (Are expressions same or different or is it impossible to tell)	% Facility: 12 years old	% Facility: 17 years old
1. $\frac{6 \times 4}{2}$, $\frac{6 \times 8}{4}$	87	91
2. $\frac{a \times b}{c}$, $\frac{a \times 2b}{2c}$	42	66

Table 4.8

Extract from Equation-Manipulating Item
from Firth's (1975) Study

Item: $P = R + S - T$

		% Facility (15 year olds)
a)	$P - R =$	50
b)	$P + T =$	76
c)	$S - T =$	29

Table 4.9

Examples of Equation-Solving Items
from Ekenstam and Nilsson's (1979) Study

Item	% Facility (16 year olds[a])
1. $\frac{30}{x} = 6$	82
2. $\frac{14}{x + 2} = 2$	58
3. $\frac{4}{x} = 3$	48

a. Sample of approximately 200 children representing the top 20% of the ability-range (Swedish) terms of mathematical structure, items 1 and 3 (Table 4.9) would appear to be comparable, with item 2 being more difficult. However, the dimension of 'psychological difficulty' results in a different ordering, with item 3 being now the most complex. Analysis of this item reveals that, unlike items 1 and 2, item 3 has a non-integer solution and is not readily solved by inspection or trial and error, procedures which can both be effectively brought to bear in handling items 1 and 2. This suggests that a possible important factor underlying children's success on a given item may be the kind of procedure or strategy that can be used to solve that problem.

The importance of the nature of the strategies available to the child, and those which may be appropriate to the solution of a given task, had in fact been indicated by Case (1974b, 1978). Working within a neo-Piagetian information processing paradigm, Case (1974b) found that older children's superior performance on a sequence completion task was due as much to the employment of a more efficient strategy as to any structural increase in processing capacity. This suggested that two tasks of similar logical structural complexity (and which should therefore require equal processing capacities for a correct solution) may nevertheless differ markedly in difficulty level if the strategies used in solving the tasks are of different orders of efficiency.

The same point was made specifically with reference to algebra by Petitto (1979). Petitto (1979) investigated the strategies that secondary school children use in solving algebraic equations, and noted two main kinds of approach which she labelled 'intuitive' and 'formal' respectively. Using the clinical interview method, Petitto examined the methods used by children to solve equations which were identical in algebraic form, but which differed in 'degree of perceptibility of embodied numerical relationships' (Petitto, 1979, p.71), for example:

$$\frac{1}{3} = \frac{2}{x+1} \qquad \text{and} \qquad \frac{14}{23} = \frac{56}{x+2}$$

Petitto found that the children interviewed solved the first of these equations by a method which was inappropriate to the second, numerically more obscure, example, namely by working in terms of the number relationships involved. Thus several of the children solved the first equation by noticing that the numerator of the right-hand-side of the equation was double that of the left-hand-side, and so doubling the denominator allowed them to conclude that x equalled 5. The children were, however, not able to apply the same approach to the second equation. Those children who solved the latter correctly did so by means of the taught equation-solving procedure. As a result of these observations, Petitto described two kinds of solution process, namely the 'formal' method which she defined as:

> "One which consists of a linear sequence of steps and which is explicitly described as a set of verbally expressed instructions or rules. In addition, the instructions or rules must apply to some general form without reference to specific instances or content."
>
> (Petitto, 1979, p.72).

and the 'intuitive' approach, defined as:

> "A process which is organised with reference to the perceived properties and relationships of the particular elements to be manipulated."
>
> (ibid, p.73)

Children's ability to handle different equations of comparable logical structure could therefore be seen to depend upon the extent to which a more 'intuitive' equation-solving procedure could be successfully applied.

<u>Child-Methods in Mathematics</u>

The possible relevance of the use of such 'intuitive' procedures for the question of children's performance in mathematics had also been noted by the CSMS project. Evidence from the transcripts of interviews conducted by the CSMS team in other areas of mathematics as well as algebra indicated that many children appeared not to use the standard 'formal' mathematical methods taught in the classroom, but rather used more informal methods of their own. These 'child-methods' (Booth, 1981a; Hart, 1981b) appeared to be essentially correct procedures which enabled the child to solve simple examples successfully. Whilst appropriate to the solution of easy problems, however, these methods appeared to be of limited applicability and would not readily extend to more complex questions or questions in which the numbers were large or non-integer. It was suggested that it was often the child's attempt to use the same informal approach in the case of the more complex problems which caused the child to make predictable errors. In the CSMS investigation of children's understanding of ratio, for example, Hart (1980c) reported that of 2257 children taking the written test, only 20 wrote down an equation of the form $a/b = c/d$ and used it correctly. During the CSMS interviews conducted on ratio it was apparent that most children used "some kind of 'building up' method which was essentially a method of addition" (Hart, 1980c, p.211). While these additive methods could be successfully applied in the case of easy examples, such as those involving ratios of the kind 1:2, 1:3 or even possibly 2:5, however, they often broke down when the problems or the ratios became more complex. At this point the children would often produce an <u>incorrect</u> addition procedure based on the additive rather than multiplicative difference between the quantities concerned (for example, enlarging something in the ratio of 3:7 would be interpreted as adding on 4 units), and thus leading to predictable kinds of error.

The possibility of differences existing between the expected method and the methods actually used by children had also been noted during the course of the CSMS investigation of children's understanding of number operations (Brown,1981b,c; Brown & Küchemann, 1976). During interviews with 12 and 13 year old children, Brown and Küchemann observed that many of the children were apparently unable to select the correct numerical expression to match a given problem such as that given in Figure 4.1. If asked to try and solve the problem, however, the children were often able to do so, but they did not do this by means of the formal mathematical process of considering the structural nature of the problem and then selecting the appropriate mathematical model (391 ÷ 23 in the case of the example illustrated), as the above finding shows. Rather they handled the problem in a more informal way which was usually based upon some kind of adding-on procedure:

YG (12 years old, in response to the item in Figure 4.1):

"You er .... I know what to do but I can't say it.
MB (Interviewer): Yes, well you do it then. Can you do it?
YG: Those are daffodils and these are flowerbeds, large you see .... OH! they're being planted in different flowerbeds, you'd have to put them in groups.....
MB: Yes, how many would you have in each group? What would you do with 23 and 391, if you had to find out?
YG: See, if I had them, I'd count them up .... say I had 20 of each ... I'd put 20 in that one, 20 in that one.
MB: Suppose you had some left over at the end when you've got to 23 flowerbeds?
YG: I'd plant them in a pot(!!)"
(Brown & Küchemann, 1976, p.15-16)

Here the method used is appropriate to a range of problems where the numbers involved are relatively small (and integer), but becomes cumbersome when the values are large or non-integer, and does not extend to questions involving algebraic terms. Attempts to apply the

A gardener has 391 daffodils.
These are to be planted in
23 flowerbeds.
Each flowerbed is to have the
same number of daffodils.
How would you work out how many
daffodils will be planted in
each flowerbed?

391-23	23÷391
23-391	391x23
391+23	23+23
23x17	391÷23

Figure 4.1 Example of item from CSMS Number Operations test. Item required children to select the operation from the list given which they would have to do in order to solve the word problem.

Item 1

In a baker's shop $\frac{3}{8}$ of the flour is used for bread and $\frac{2}{8}$ of the flour is used for cakes.

What fraction of the flour has been used?

Item 2

$\frac{3}{8} + \frac{2}{8} =$ ....

Figure 4.2 Two items from CSMS Fractions test.

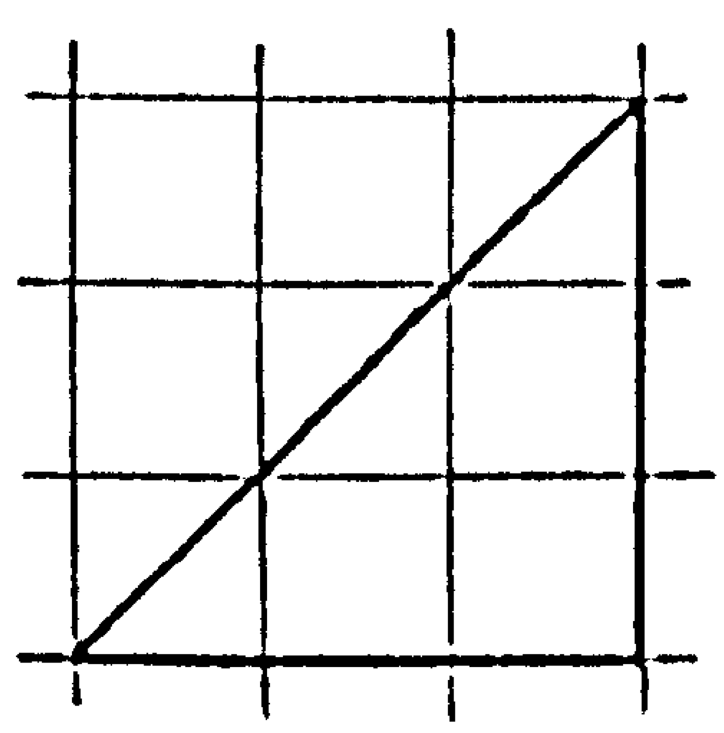
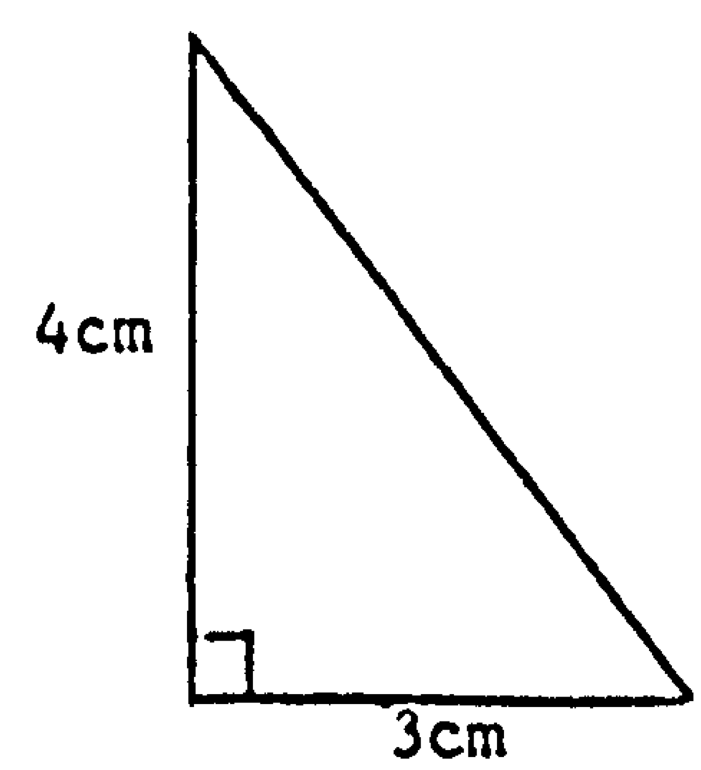
same method in such cases might be expected to lead to considerable difficulty.

Other examples of the possible use of informal methods on easy items could also be inferred from the CSMS written test data. Thus while 77 percent of 15 year olds could successfully solve the problem in item 1 in Figure 4.2, only 68 percent correctly answered the corresponding computational item (item 2, Figure 4.2), showing that some children at least used methods other than the standard fraction algorithm to solve the word problem. Similarly, the difference in facility between the two CSMS items shown in Table 4.10, where the child was asked to find the area of each shape, indicated that some children who were able to solve the first item were not able to use the formal 'rule for area of a triangle' which was needed in order to solve the second item.

In general, it was considered that the CSMS research provided strong indications that "children frequently tackle mathematics problems with methods that have little or nothing to do with what has been taught" (Küchemann, 1981, p.118), and that "although children were able to use a strategy which was at least roughly correct, they did not recognise how the problem could be symbolised" (Brown, 1981c, p.29). This issue concerning the nature of the procedures which children adopt in solving mathematical problems may be one of considerable importance. In particular, it can be seen that the use by the child of non-standard methods may be expected to lead to especial difficulty in algebra, where the formal representation of problem structure is often required. In so far as generalised arithmetic may be regarded as "the writing of general statements representing given arithmetic rules and operations" (page 1 of this thesis), then the possibility of non-use and non-recognition of these

Table 4.10

Example of Two 'Area' Items from CSMS Measurement Test
Differing re Nature of Possible Solution Strategy

Item	Area of: 	Area of: 
% Facility		
(13 year-olds)	87	38
(14 year-olds)	92	47

rules and structures in arithmetic itself may have considerable consequence for children's performance in algebra. For example, if a child does not recognise that the total number of elements in two sets of 23 and 17 elements respectively can be represented by 23+17, then the chances are perhaps slight that the child will readily appreciate that x+y represents the total number of elements in the case of two sets containing x and y elements respectively.

Points for Investigation

Summary

In summary, it seems that any investigation into children's understanding in elementary algebra, and particularly into the reasons for the consistent errors which they may make, must take into account at least two main factors. Firstly, it seems that children may interpret and use letters in a variety of ways which are not always anticipated by teachers, and which may influence the child's ability to successfully handle algebraic problems. Children may avoid viewing letters as numbers altogether, either by ignoring the letters or by treating them as objects. Even where letters are regarded as numbers, they may be 'removed' by immediate numerical substitution, or at a more sophisticated level, may be regarded as 'specific unknowns', their possible role as generalised number or variable not being recognised. In addition, the possible significance of alphabetical sequencing for the values ascribed to letters, and the possibility that children equate changes in variable symbol with changes in referent value, must also be taken into account. In investigating children's errors in generalised arithmetic, therefore, it was decided that the likely involvement of level of letter interpretation must receive careful and particular attention.

Secondly, while degree of (logical) structure complexity of an item may have implications for a child's success on that item, it seems that this effect may operate by virtue of the kinds of procedure which the child brings to bear in handling that problem. Consequently, it was thought necessary, in the research adopted here, to investigate this issue of the use of 'child-methods' as well as that of level of level interpretation.

The Present Research

The Strategies and Errors in Secondary Mathematics investigation into the causes of errors in generalised arithmetic identified by the earlier CSMS project therefore commenced within the framework of two 'hypotheses':

(1) the errors observed depend (in part) on the child's interpretation of the letters involved;

(2) an error may also arise as a consequence of the procedures that the child uses in solving arithmetical problems of a similar kind.

The precise errors investigated were those which had been found by the CSMS research to have a particularly high incidence, and which were summarised in Table 4.6 earlier in this chapter.

The following four chapters describe the research programme itself, beginning with the 'ascertaining' experiment, and then presenting details of the teaching experiment phase, in terms of the initial small group investigation and the subsequent re-trial of the modified teaching programme and the final whole class verification. The ascertaining experiment itself comprised two phases, namely (a) a preliminary round of interviews designed to elaborate more specific hypotheses (within the framework described above) concerning children's difficulties in algebra, and (b) a 'follow-up' round of interviews designed to test and clarify these hypotheses further.

Part 2

THE RESEARCH PROGRAMME - METHOD AND RESULTS

CHAPTER 5: THE ASCERTAINING EXPERIMENT - PHASE ONE

The aim of the interview phase of the research was to analyse those specific errors in algebra identified by the CSMS team, in order to obtain information on the conceptions and strategies upon which these errors are based. The approach selected as appropriate to this objective constituted individual interviews with children identified as making the errors under study. The interviews were conducted in two phases. Phase one, the subject of this chapter, was designed to enable specific hypotheses (within the framework already established by consideration of the CSMS results) concerning the causes of the observed errors to be derived. Phase two (see Chapter 6) was designed to test and elaborate those hypotheses further.

Description of the Interview Sample

Selection of Schools

Five comprehensive schools, representing as varied a sample as possible with respect to such features as ability-range and socio-economic background of students, geographic area (within Greater London), and school size, type and organisation, agreed to participate in the programme. Assessment of each school's description with regard to such factors as ability-range and socio-economic level was made in general terms only, and on the basis of information provided by individuals familiar with the schools and their environments (such as teachers, teacher-trainers and other researchers), and from information provided by the schools themselves.

Four of the five schools were used in phase one of the interviews, the remaining school being included only in the second phase because of internal conditions at the time of the first phase of interviews (the school was holding examinations). A brief description of each school is included in Table 5.3 entitled 'Phase One Interview Sample'.

Selection of the Sample for Testing

Following the procedure adopted by the CSMS Algebra testing programme, attention was confined to students in the second, third and fourth years of secondary schooling (ages 13 to 15 years) in the first instance.

A total sample size for the phase one interviews of approximately 50 students, or approximately 16 students from each year group under study, was decided upon. This apparently arbitrary figure was arrived at as the result of a compromise between the desire to base the research upon as large and representative a sample as possible, and such considerations as (i) the requirements of economy of researcher time (each half hour of interview requires approximately two hours of transcribing), (ii) feasiblity of analysis (it is as difficult to extract essential information from too much interview material as from too little), and (iii) the need to delimit intrusion into each participating school.

In order to obtain a total sample of this size from the four schools used in phase one, it was therefore sufficient to select approximately four students from each of the three target year groups in each school. This was done by administering the CSMS Algebra test to half a class (approximately 16 children) in each year group in each school. Since it was intended to interview (at least initially) only children who were making the identified errors, and since the CSMS data had indicated that approximately 40 per cent of each age group were in fact making those errors, such a procedure was expected to generate a set of potential interviewees of the required number. The classes chosen for participation in the testing were selected at random from the full year group in the case of schools with mixed ability classes in mathematics, and from the middle ability groups in those cases where the children were streamed or banded for mathematics. In the latter event, the choice of middle ability groups as the sampling base was made in order to ensure the identification of a sufficient number of interviewees, since higher ability groups would presumably contain fewer children making the identified errors, and to permit selection of children for interview who would be reasonably confident in explaining their methods and ideas, which may be less true were the selection made from lower ability groups.

Half of each class thus selected was given the CSMS Algebra test, the remaining half being presented with the CSMS Ratio test, as a prelude to the investigation of children's errors in ratio which also formed part of the SESM programme. The Algebra and Ratio tests were given to alternate children, in order to randomise their allocation and to reduce the likelihood of cooperative efforts in completing the tests.

The CSMS Algebra Screening Test: Administration and Results

The CSMS Algebra test was used as a screening test in order to permit identification of children making the errors under study. The test was presented in its original form (see Appendix 1A for examples of items) and according to the administration procedures outlined by CSMS (Küchemann, 1980).

In essence, this involves the administration and checking of two trial items, the exhorting of children to make an attempt at as many items as possible, and the completion by the pupils of the test paper under examination conditions, but without the imposition of a time limit. The time generally taken to complete the paper was approximately 25 minutes. The test was administered in each case by the class teacher and marked by the experimenter. The marking scheme used was essentially similar to that used by CSMS (see Hart, 1980b, pp. 178-180), except in so far as some answer-categories were combined. The marking scheme as thus amended for use in the present study is given in Appendix 2A, together with a brief rationale for the conjoining of answer-categories thereby effected.

All items on the CSMS Algebra screening test were coded for correct and various given categories of incorrect answers. Consideration of the pattern of correct answers permitted the allocation of one of four 'levels of understanding' to be ascribed to each individual, according to the criteria described by CSMS (Küchemann, 1980). Children who had not met the criterion for Level 1, the lowest of the four levels, were ascribed to Level 0. As a sampling check, the distribution of levels of understanding in the total sample from the four schools tested in phase one was compared with the data for the sample of 3550 children in the CSMS algebra programme. It was expected that in the case of the second and third year groups (aged 13 and 14 years respectively) the fit should be a close one, since the majority of classes sampled by SESM in these year groups were mixed ability and could therefore be regarded as being drawn from the same ability range as were the children tested by CSMS. In the case of the fourth year group, however, all four schools used in the present study employed a streaming or banding method of organisation for mathematics, so that the policy of sampling only from middle ability groups would be expected to result in a decrease in proportion of children at the highest and lowest levels of understanding, and a consequent enhancement in proportion observed at the middle levels. Both of these expectations were fulfilled. The closeness of fit of the SESM test sample with the CSMS total sample (allowing for the expected departure in the case of the fourth year group) enabled these two samples to be regarded as being drawn from the same population (see Table 5.1).

The incidence of 'level misplacements' (children who performed at a higher level of understanding, as defined by the CSMS criteria, without successfully mastering all lower levels) was 1.9 percent for the second year group (aged 13), 2.9 percent for the third year group (aged 14) and 0 percent for the fourth year group (age 15). These figures are well within the seven percent limit for scalability of levels set by the CSMS team (Hart, 1980b).

Selection of Children for Interview

In terms of the interpretation put forward by Küchemann (1981), inspection of the error responses for selected items indicates that these errors appear to be based upon:

a) the conjoining of numerical and algebraic elements, suggested by Küchemann to reflect the child's treating the letters as objects which may simply be collected up (e.g. items 2, 3 and 7 in Table 4.6).

b) the handling of algebraic items by ignoring the letters or by numerical substitution (e.g. items 4, 5 and 6 in Table 4.6).

c) the viewing of letters as specific unknowns rather than generalised number (e.g. item 8, Table 4.6).

Answers which appeared to relate to each of these kinds of letter usage were coded separately and the incidence of each code-type was recorded for the test as a whole, including other items as well as

Table 5.1

Comparison SESM Test Sample and CSMS Large-Scale Algebra Sample in Terms of Percentage of Children at Each Level of Understanding[a]

CSMS Algebra Level	Year Group					
	2nd (Age 13)		3rd (Age 14)		4th (Age 15)	
	SESM	CSMS	SESM	CSMS	SESM	CSMS
0	13	10	7	6	2	5
1	53	50	44	35	18	30
2	19	23	29	24	48	23
3	13	15	17	29	25	31
4	2	2	3	6	7	9

a. Percentages based on a sample of 167 children for SESM and 3550 for CSMS those of particular concern to the study. Children producing an error answer of the appropriate kind to a given number of items (the criteria are given in Table 5.2) were selected as being eligible for interview. A total of 48 children were consequently interviewed in phase one, these being distributed among the four schools as indicated in Table 5.3.

The Phase One Interviews

As a basis for the individual interviews to be held with children making the errors under study, a set of questions comprising parallel items to the CSMS 'error' items and some extension items was prepared. The use of alternative items for interview discussion rather than the original testpaper items was considered preferable for a number of reasons. Firstly the requirement that children now perform a similar item to one previously answered incorrrectly permits an assessment, albeit crude, of the stability and generality of the error strategy or concept concerned, and enables errors due to particular idiosyncratic features of the original item to be identified. Secondly, the inclusion of extension items permits more specific exploration of item-type, in terms of the effects of particular variations in item and the points at which given errors may arise or be abandoned, and also allows for the removal of distracting features of an item where these are considered non-relevant to the issue being investigated. Thirdly, the presentation of items different to those on the CSMS test reduces the likelihood of the children being distracted by questions relating to their earlier test performance, and likewise minimises the chances that children will try to remember what they had done previously instead of explaining what they are currently doing - it being considered that children's description of remembered activity is

Table 5.2

Criteria for Selection of Children for Interview

Indication	Number of CSMS Items	Number of Errors	Number Interviewed per Category
1. Letter as object	13	5	48
2. Letter ignored	11	3	22
3. Alphabetic Substitution	13	3	7
4. Letter as specific unknown	2	1	34

Table 5.3

Phase One Interview Sample

School	School Type (Location)	Number of Children Interviewed 2nd yr (Age 13)	3rd yr (Age 14)	4th yr (Age 15)	Total
A	Boys Comprehensive (Surrey)	4	4	5	13
B	Girls Comprehensive (N. London)	3	5	3	11
C	Co-ed. Comprehensive (S. W. London)	4	4	5	13
D	Co-ed. Comprehensive (Middlesex)	3	3	5	11
	Total:	14	16	18	48

less reliable than that of current activity. Finally, presentation of separate items prevents the risk of distraction by other (non-target) items as may occur if the whole test paper were presented in interview discussion. In line with this latter consideration, each item used in interview was introduced individually by being presented on a separate card, a procedure which also permitted changes in sequence of presentation or of actual items submitted, depending on individual children's responses. The set of items available for use during the interviews is given in Appendix 2B, together with a brief rationale for inclusion in the case of amended or extension items.

Design of Interview Schedule

When a child produces an incorrect answer to a problem, the error resulting in that answer may have occurred at one of several points in the process of solving the problem. The children may have mis-read the question, or may have mis-interpreted the question or part thereof. Alternatively, despite a correct interpretation, an incorrect method may have been used in solving the problem, or, finally, the answer may have been wrongly encoded. There is, of course, the possibility of any combination or interaction of such errors. For example, consider the child who gives the answer 5e2 or e10 or 10e to the item requiring the area of a rectangle measuring e+2 by 5. This wrong answer may have resulted because the child:

a) interpreted the letter as a 'thing' which could be merely collected up with the numbers (an interpretation or 'input' error);
b) did not know how to interpret the letter <u>or</u> did not know how to operate with it and so performed the numerical calculation and wrote down the letter afterwards (an

input/process error);

c) knew that 'e' represented a number expressing part of the length of the base, but thought that area meant multiplying everything together (a process error);

d) interpreted the letter correctly, applied the correct method, but recorded the 'answer' to 'e added to 2' as e2 or 2e (an output error).

There may, of course, be other possibilities. The consideration of an input-process-output model of this type permits a clearer picture to be obtained of the point(s) at which the child's understanding of the problem breaks down. An interview schedule designed to separate out these components of the child's problem-solving process and based upon the error-analysis approach elucidated by Newman (Newman, 1977; see also Casey, 1978; Clements, 1980) was thus developed for use in individual interviews. An outline of the schedule so developed is given in Appendix 2C.

In order to allow the child freedom in following and explaining his or her own train of thought, and to permit the interviewer flexibility in pursuing other aspects of interest, the interview schedule was not rigorously adhered to. Nor was it appropriate to ask every question of each child on every item presented. The schedule was used rather as a framework to give a general structure to the interview, and to indicate the overall direction that each discussion should take. Regardless of individual variations in interview, however, care was taken in every case to ascertain the child's interpretation of any item presented, and the child's way of encoding each answer.

Attention was also paid throughout the interviews to the kind of strategy that the child brought to bear in handling the problems. Of the list of possible strategies in mathematical activity outlined in chapter 2, it was considered unlikely that the strategies of random or systematic trial and error would be observed, in view of the nature of the items selected for investigation in the present study. The 'search for pattern' strategy was similarly considered unlikely to occur. Consequently attention was largely confined to the possible involvement of the following three strategies indicated in chapter 2:

IS - informal ('intuitive') strategy

SA - search for algorithm (i.e. taught rule)

RFS- recognition of formal structure of problem and the consequent selection of the appropriate procedure.

In view of the indications in the CSMS data, as well as in the work of researchers such as Petitto discussed earlier (Chapter 4), of the possible use of 'child-methods' and the likely implications of this for children's performance in algebra, interest was particularly focussed on the possibility of employment of informal or 'intuitive' strategies in children's responses to the interview items.

The Interviews

Each child was interviewed individually by the experimenter on a subset of the interview items (see Appendix 2B) and according to the interview schedule outlined in Appendix 2C. The particular items given to any child depended upon that child's responses to the target items on the CSMS algebra screening test, and to the items already presented in interview. Each interview lasted approximately 30 minutes and was tape-recorded and later transcribed. In addition, the child was encouraged to show any workings-out on a sheet of paper, and also asked to record the final answer. The experimental data thus comprised the interview transcripts and the children's written scripts.

Analysis of Interviews

The verbatim interview transcripts were examined by the experimenter in conjunction with the written scripts from the interviews. After several readings of these, a summary of each transcript was prepared. This set of summaries was then closely examined, and a list of salient points suggested to describe the various areas of difficulty revealed by the interviews was extracted. As a check, a random sample of interview transcripts was given to each of two other researchers working with the SESM project, and their summaries and list of points compared with those of the experimenter. The number of children demonstrating each kind of difficulty was ascertained per item and per interview, and individual response-profiles with respect to selected areas of difficulty were also derived, in order to determine both the consistency of reponse, and whether certain constellations of response were found to be associated.

Findings

The difficulties children have in algebra which were revealed by the interviews appeared to relate to three main areas, namely the meaning children attached to letters, the process of operating with letters, and questions of notation and convention in algebra.

Meaning attached to letters. A summary of the points derived from the interviews with regard to this area of difficulty is given in Table 5.4. Illustrations of each of

Table 5.4

Difficulties Identified in Interviews - Meaning Attached to Letters

	Points Derived from Interview	Number of Children[a]	Proportion[b]
1.	Letter as representing number confused with letter as representing object, usually in 5x + 8y type example.	11	0.48
2.	Different letters represent different numbers (as converse of 'same letter represents same number', also as result of experience).	26	0.74
3.	Letters represent whole numbers.	6	0.75
4.	There is a 'pattern' in the relationship between letters and the numbers they represent (result of experience with 'codes'?), e.g.		
	a) x, y, z → 3, 4, 5 or 10, 20, 30 etc.	3	0.10
	b) y is 'higher' than p	3	0.10
	c) fixed alphabetic substitution.	4	0.13
5.	Meaning of letter ignored, occurring in 'abstract' examples (3 + 5y; 2x + 8y + 3x etc.) i.e. such problems treated as mere manipulation of symbols and 'rules' invented to govern manipulation:		
	a) add up all the numbers, then put down each letter that occurs (once only)	18	0.58
	b) add up all the numbers, then put letter for every time it occurs in expression	4	0.13
	c) add up all the numbers, then put down 'highest' letter i.e. one that occurs most often.	1	0.03

a. 'Number of children' refers to the number of individuals demonstrating a given conception or procedure etc. at any stage of an interview.

b. 'Proportion' represents the ratio of that number of children to the total number of children receiving item(s) or questioning relevant to such demonstration.

these points can be taken directly from the interviews themselves. For example, in discussing the meaning of 'y' in 'add 3 onto 5y', or of 'x' and 'y' in simplifying 3x+8y+2x:

(Interviewer: I; Benedict: B, 14 years)

I: What does 'y' mean (in 'add 3 onto 5y')?
B: It can mean anything.
I: Like what?
B: Pineapples, grapes........
I: So it could mean something like pineapples?
B: Yes.

(Julie: J, 14 years)

I: (Re simplifying 3x + 8y + 2x.) Do the x and y mean anything there, do they stand for anything?
J: No, they're just letters, you have them in algebra.

In some instances this view of letters does lead to a workable rule for simplifying algebraic expressions:

(Interviewer: I; Leon: L, 15 years)

I: (Discussing an answer of 5q - 3w to an alternative item invented during the interview.) What do the q and w mean?
L: Well, yes, it could be various things.
I: So what could the q be ....
L: It could be ... 5 bananas minus 3 apples, something like that.
I: I see. Could it be anything else?
L: Yes! It could be anything - chairs, tables ...
I: And is that why you can't take them away?
L: Yes, they aren't the same <u>sort,</u> you can't take them away.

However, this is not to be relied upon:

(Tristan: T, 15 years)

T: (Simplifying 5x + 8y to 13 xy.) Well, then, I could write that as 13xy.
I: Could you?
T: Yes, well, what I've done is, I've got the 8 blankets (y interpreted as blankets) and just the 5 sweets (x interpreted as sweets) on them.
I: And does that give you 13xy?
T: Yes, 13 altogether. 13 of ... xy. And the same for the 2qw (another item).
I: And is that what the q and w mean, like sweets

and blankets?
T: Well, anything really. I'm just giving that as an example.
I: What else could you give as an example?
T: Well, desks and chairs. Cards, buttons, houses..
I: Oh, I see, all sort of ...
T: Yes, things.
I: Coming back to this one (number of goals scored by West Ham and Manchester United), what was the x and y standing for there?
T: Goals. Amount of goals.
I: And here ($x + y + z = x + p + z$)?
T: Well, anything again. y could be ... a tape recorder. And p a tape. Different things.

At a more sophisticated level it is clear that, while representing numbers, different letters do not represent the same value. This is illustrated by students' responses to the question as to whether $x + y + z$ can ever equal $x + p + z$ and is considered to reflect children's views of letters as specific unknowns (i.e. as representing a particular unique although as yet unknown value) rather than as generalised number, i.e. taking on a range of values (see also Table 5.5). For example:

(Interviewer: I; Trevor: T, 15 years)

T: It won't be true, never.
I: Never?
T: Never, because, it'll have different values... because p has to have a different value from y and the other values, so that'll never be true.
I: So p has to have a different value ... why do you say that?
T: Well, if it didn't have a different value, then you wouldn't put p, you'd put y. You see, you put a different letter for every different value.

(John: J, 13 years)

J: (In response to the suggestion that p <u>might</u> equal y). ... it's a bit pointless because you probably wouldn't have that.
I: Wouldn't have what?
J: Wouldn't have down ... two letters for one number.

(Mandy: M, 15 years)

Table 5.5

Meaning Attached to Letters -
Number of Children Interviewed Giving Identified Responses

Item $x+y+z = x+p+z$ True: always never sometimes, when	Total number of children given item		35
	Answer correct		8
	Letter as specific unknown:		
	Different letter means different values	14	
	Other	9	
	Letter as specific unknown: Total		23
	Literal reading		1
	Sum (values) required		3

M: But if y and p were the same, you'd have thought they would have put x, y and z, instead of x, p and z.

(Tristan: T, 15 years justifies his opinion that y and p are different values):

T: ...y couldn't be the same as p.
I: Oh, I see, so using different letters...
T: Means they're different amounts.
I: Oh, I see. And are they always different amounts?
T: Well, I've always found they're different. I've never come across one where they're the same.

The number of children giving this kind of response on interview is given in Table 5.5.

Operating with letters. The difficulty concerning the process of operating with letters is less explicitly expressed and became apparent mainly from a consideration of the analysis of responses to particular items (see Tables 5.6, 5.7 and 5.8). This difficulty appeared to be compounded of several aspects, including the use by children of informal approaches to the solution of items, and their non-recognition of the formal structure of the problems presented. Thus children's errors in this regard related to the choice of a correct but inappropriate method which could not be adequately symbolised in algebraic terms, the inability or perhaps unwillingness to record an algebraic statement (possibly linked to an assumption that what is required is a specific numerical answer), and a possible absence of an operational model in arithmetic itself, so that generalisation to the algebraic expression was perhaps unlikely. These ideas were only tentatively broached at this stage and required further investigation in the ensuing phase of interviewing.

Consideration of responses to two of the 'perimeter' items (Table 5.7) may illustrate some of these points. Whilst all the children given item (a) in Table 5.7 were able to describe a correct method for

Table 5.6

Difficulties Identified in Interviews - Operating with Letters

Points Derived from Interview	Number of Children[a]	Proportion[b]
1. When operation (method) involved is firmly established, letters appear to present relatively little problem. Children for whom the operation is itself unclear have great difficulty in handling letters (e.g. some children, while adding 2 and 5 and getting 7 do not represent what they have done as 2 + 5, but either 'know' the outcome, or use column addition).	18	0.56
2. Using letters - 'I can't do it because I don't know what the numbers are'. (Assumption that what is required is an actual answer; not understood that an 'answer' can be an expression)	17	0.46
When pushed, will fall back on:		
a) measuring to get value	13	0.35
b) attempt to handle by assuming a particular value and hence getting an 'answer'	6	0.16
c) attempt to handle by alphabetic substitution.	3	0.08
3. Successful approaches to the handling of letters included:		
a) reference to a specific numerical example, e.g. 'if x were 2 and y were 3 I'd add, so x + y'	6	0.17[c]
b) a more generalised version of 'add whatever number x is to whatever number y is'	29	0.83[c]

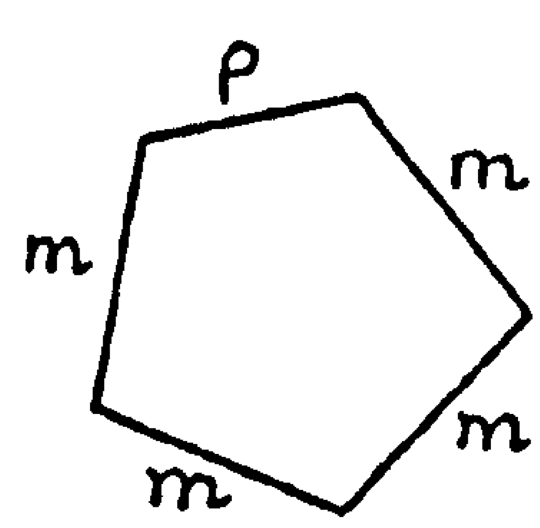
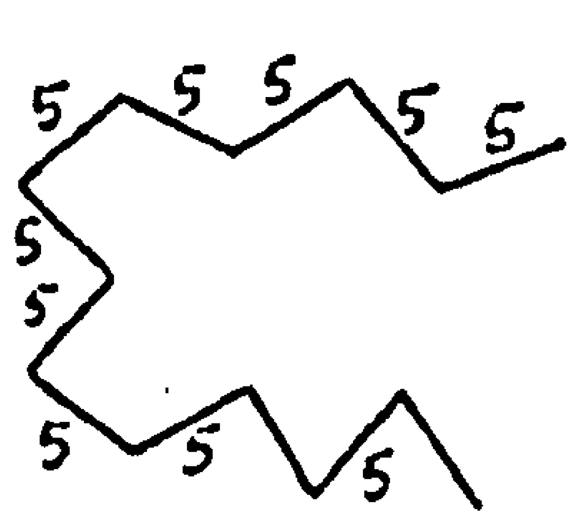
a. 'Number of children' refers to the number of individuals demonstrating a given conception or procedure etc. at any stage of an interview.

b. 'Proportion' represents the ratio of that number of children to the total number of children receiving item(s) or questioning relevant to such demonstration.

c. Proportion represents the ratio of the number of instances of use of the indicated strategy to the total number of successful approaches, e.g. 0.17 of successful solutions were obtained by reference to a specific numerical example.

Table 5.7

Operating with Letters (Perimeter Items) -
Number of Children Interviewed Giving Identified Responses

Item	Response		
a) Perimeter of :	Total number of children given item		28
	Answer correct, including:		
	Initial confusion powers	7[a]	
	Ambiguity 4m and 4m's	6[a]	
	Answer unsimplified (m + m + m + m + p)	8[a]	
	Answer correct: Total		20
	Method correct, answer incorrect, including:		
	Unconventional recording	2	
	No symbolic recording	4	
	Method correct, answer incorrect: Total		6
	'Primitive' method (Count up, measure)		2
b) Perimeter of:	Total number of children given item		13
	Answer correct		1
(p sides)	Method correct (multiplication), answer incorrect, including:		
	Inappropriate (column) recording	1	
	No symbolic recording	4	
	Method correct, answer incorrect Total		5
	'Primitive' meth[illegible], count up)		7

a. Numbers total to more than 20 since some children's answers included more than one aspect.

finding the perimeter of shapes, two children (out of 28) revealed a more informal and 'primitive' counting-on approach which necessitated the provision of numerical data and could not be related to perimeters given in algebraic terms:

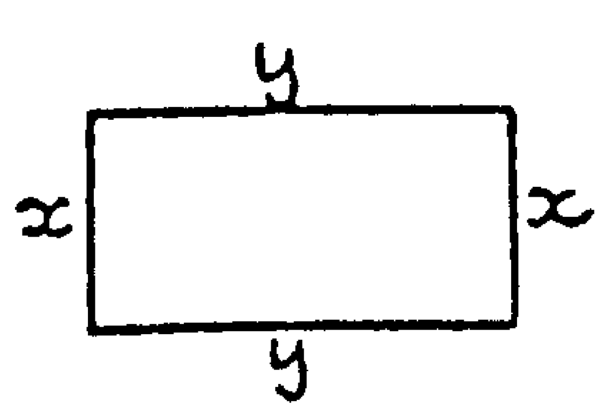
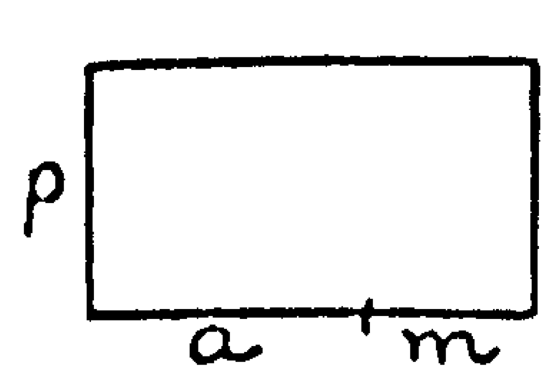
(Interviewer: I, Maureen: M, 14 years)

I: (After M has been unable to suggest anything for the perimeter of the shape with four sides of length m and one side of length p.) You know how to find it for this shape? (A new shape B provided with sides 5,5,4,3 and 2.) Right? What did you do there?
M: Counted all the numbers up.
I: Right. What do you think you'd do here?
M: Can't, it's got no numbers.
I: If there were numbers you could do it?
M: Yes. But not if it's got no numbers. I don't know what m means. And what p is.
I: Suppose I tell you, that m just means some number and p means a number, but it's a game and you don't know what the numbers are. Could you tell me anything about how you'd find the perimeter, what you'd do to the m's and p to get the perimeter?
M: I suppose I'd have to measure them or something.
I: And then what?
M: Count them up.

Item (b) in Table 5.7 is more interesting in this regard, in that seven out of 13 children now used a 'more primitive' adding-sides method which was inappropriate to the correct algebraic expression of the perimeter, which now required that the child recognise that perimeter can be expressed in this kind of item by the product of side length and number of sides. The same Table shows that four out of 28 (item (a)) and four out of 12 (item (b)) children gave a correct verbal description of the required method in each case, but were unable or unwilling to produce a written symbolic representation of that method as an answer. This apparent reluctance to produce an algebraic expression under such circumstances was noted also in response to other items. For example, Table 5.8 shows that nine out

Table 5.8

Operating with Letters (Additional Items) -
Number of Children Interviewed Giving Identified Responses

Item			
	Total number of children given item [a]		28
1. West Ham scores x goals, M.U. scores y. Total goals scored?	Answer correct		10
	Method correct, answer incorrect, including:		
	Inappropriate (column) recording	3	
	Conjoined answer	8	
	Algebraic closure (e.g. $x+y \rightarrow z$)	1	
	No symbolised recording	9	
	(also no recording with numbers)	(8)	
	Method correct, answer incorrect: Total		18
2.a) Area of: 	Total number of children given item		34
	Answer correct, with generalisation from numerical example	8	
	Answer correct: Total		31
	Method correct, answer incorrect, including:		
	No symbolised recording	3	
	Method correct, answer incorrect: Total		3
	Total number of children given item		28
b) Area of: 	Answer correct by means of alternative method (brackets not required)	1	
	Answer correct: Total		2
	Method correct, answer incorrect including:		
	Brackets omitted, otherwise correct	10	
	Conjoined answer	7	
	No symbolised recording	6	
	Method correct, answer incorrect: Total		26
	Note: Other unconventional recording	3	

a Evidence of search for a 'final answer' (i.e. $x + y$ not perceived as a final answer) was observed in 19 cases.

of 28 children in the West Ham item, and three out of 34 and six out of 28 in the respective area items, did not produce an algebraic expression at all. In the case of the West Ham item, eight of the nine children who did not produce an algebraic expression also did not symbolise their method in a subsequently given arithmetic example (West Ham scores 3 goals and Manchester United scores 2 goals, how many scored altogether?). These children merely wrote down the replacement value, 5, remarking that there 'is nothing to put, you just add'. One child who was considering larger numerical values (24 and 25) resorted to the column method of recording additions. A further five children who also wrote down only the replacement value in the arithmetic case, did produce an algebraic answer in the x + y example, but the answer produced was the (incorrect) 'replacement value' xy.

Notation and convention. Difficulties relating to the way in which algebraic expressions are symbolised are summarised in Table 5.9. Two of these difficulties, namely non-use of brackets and the replacement of an open algebraic sum (e.g. a + m) by the conjoined term (am), have already been indicated in Table 5.8. Table 5.10 gives the proportion of children producing a conjoined answer in each of three items to which such a response was possible.

In some cases the conjoined term is seen as necessarily equivalent to the unclosed sum:

(Interviewer: I; Mark: M, 13 years)

M: (Explaining his answer of k6 to the item 'multiply k+2 by 3'.) Well, I thought ..... k+2 was the same as k2.
I: Oh, I see, so k2 is the same as k+2?
M: Yes.

(Michael: MM, 14 years)

Table 5.9

Difficulties Identified in Interviews - Notation and Convention

Points Derived from Interview	Number of Children[a]	Proportion[b]
1. Addition/'putting together' with letters, e.g. a + m recorded as am, x + y as xy, k + 2 as k2.	17	0.49
2. Meaning of 4m, xy:		
a) ambiguity re 4m as '4 m's', not necessarily recognising this as equivalent to 4 x m	11	0.52
b) confusion '2 lots of x $\longrightarrow$ 2x' and '2 lots of 7 $\longrightarrow$ 27' written as 2_7 to distinguish	5	0.31
c) xy: if x = 3, y = 2, xy $\longrightarrow$ 32.	5	0.29
3. Equivalence of m + m + m + m and 4m not recognised (this equivalence often not clear even in case of numbers).	10	0.32
4. Confusion with powers (m + m + m + m $\longrightarrow m^4$).	8	0.26
5. Brackets - need for use not appreciated	15	0.88
a) first-written operation is performed first	9	0.53
b) when child 'knows' which operation to do first, sees no need to record fact.	9	0.53

a. 'Number of children' refers to the number of individuals demonstrating a given conception etc. at any stage of an interview.

b. 'Proportion' represents the ratio of that number of children to the total number of children receiving item(s) or questioning relevant to such demonstration.

Table 5.10

Notation and Convention (Conjoining) -
Number of Children Interviewed Giving Identified Responses

Item		
a) Area (a + m)x p	Total number of children given item	28
	Conjoined answer am	7
b) West Ham x + y	Total number of children given item	28
	Conjoined answer xy	8
c) Multiply k + 2 by 3	Total number of children given item	14
	Conjoined answer k2	5

MM: (In response to the 'area of rectangle p by a+m' item.) Well, I suppose I could put it down like am times p (writes am x p).
I: Oh, right, now what's that am bit?
MM: That is really the joining together .... of the two lengths.
I: Oh, I see, but how would you get that joining together, what would you do?
MM: First I'd have to measure what the length of a and m is, and then I'd add them together.

Whilst in other cases the conjoined term seems to be more the outcome of a need to replace an unclosed sum by a single element answer, as was also apparently the motivation (together with an attempt to make some sense out of an otherwise apparently nonsensical situation) underlying MM's decision to use 'algebraic closure' on the West Ham and Manchester United item:

(Interviewer: I; Michael: MM, 14 years)

MM: I can't see any way really, I mean, x and y, it's really hopeless. Unless, of course, you put above it x = 3 or y = 2. I've seen that sometimes. And then they put down the question, and you put down the answer.
I: If you did know what the x and y were, what would you have to do to get the answer?
MM: I'd have to add the y goals onto the x goals, and that would be, say, z goals.
I: So you'd say .... y .....
MM: Plus x. Yes, so therefore, if they left it like that, I suppose I'd better put down the 'z' really (writes just 'z').

Other children appeared to have achieved some measure of recognition of the conjoined term as a product rather than a sum, but appeared to now transfer this interpretation also to the case of numerical terms where these appeared in the same context:

(Interviewer: I; Tanika: T, 15 years)

T: (Regarding the perimeter of the shape with sides labelled 3, 7, 7, x and x.) I'd do, two sevens, two x, and a one three ... (writes 27+2x+13).
I: Now, look what you've written down there. Two seven. Now what do you mean by that?
T: Oh! It should be smaller! 'Cos otherwise you'll think ....(writes 2_7, i.e. writes the 7 as a subscript).

I: Right, what do you mean by that?
T: Two sevens, multipy it, 'cos even if you add it, it's still going to be 14, isn't it?
I: OK, so you mean.....
T: Two multiplied by seven, two multiplied by x ....
I: OK, so you don't mean twenty-seven?
T: No! Two lots of seven. Oh ... (writes 1_3 instead of 13).

Table 5.11 presents two further items to which a type of conjoined answer was observed (e.g. $3x + 8y + 2x \longrightarrow 13xy$), but as this appeared in these instances to reflect the operation of a 'rule' which stated 'add all the numbers and then put down the letters' rather than an explicitly additive conjoining, the data have been presented separately. The same kind of 'rule' appears again in response to the item 'multiply k+2 by 3', where six out of 14 children describe their working in these terms (see Table 5.12), and this item also indicates a proportion of children (five out of 14) whose otherwise correct answer is rendered incorrect because of their apparent non-recognition of the need to use brackets, a point also illustrated in Table 5.8 (item (b)) where 10 out of 28 children may be likewise categorised. Indeed, in both items only one child (out of 14 and 28 respectively) made use of brackets in recording their answers, although the need to perform the addition <u>before</u> the multiplication in each case was often clearly stated:

(Interviewer: I; Neil: N, 15 years)

N: (Re item requiring area of rectangle measuring a+m by p.) p times ... a plus m (writes p x a+m).
I: Right, so you've written down p x a+m. And what would you actually do, what would you need to do first?
N: I'm not with you.
I: Right, why did you say p times a plus m?
N: Because you're timesing that side (p) by that side (a and m), and that side (a and m) you can't do, so you've got to add that (a) onto that (m), to times the two sides together.
I: Right, so which bit would you do first?

Table 5.11

Convention/Meaning Attached to Letters -
Number of Children Interviewed Giving Identified Responses

Item		
a) Add 3 to 5y	Total number of children given item	24
	Answer correct	8
	Answer incorrect - 8y (add numbers and put down letter)	15
	Value of y required	1
b) 3x+8y+2x	Total number of children given item	28
	Answer correct, explanation given 7 Answer correct: Total:	18
	Answer incorrect - 13xy	7
	Values required	3

Table 5.12

Notation and Convention (Brackets) -
Number of Children Interviewed Giving Identified Responses

Item			
a) Multiply k+2 by 3	Total number of children given item		14
	Answer correct		1
	Method correct, answer incorrect, including:		
	Brackets omitted, otherwise correct	5	
	Use of rule 'place 3 in front'	1	
	Method correct, answer incorrect: Total		7
	Method incorrect (multiply numbers, put down letter)		6
	Note: Conjoined answer included somewhere in working	5	
b) Area a + m by p	Total number of children given item		28
	Answer correct (using brackets)		1
	Answer correct (brackets avoided)		1
	Method correct, answer incorrect, including:		
	Brackets omitted, otherwise correct	10	
	Conjoined answer am	7	
	Unconventional recording	3	
	No symbolic recording	6	
	Method correct, answer incorrect: Total:		26

N: ...I'd add those two up (a and m), and then I'd times it by p.
I: And is that what you've written?
N: Yes
I: Suppose I said I thought that (p x a+m) meant p times a. And then plus m.
N: Oh no, it can't be that. If you did p times a, you'd only get a bit of it (area). You've got to do the a plus m to get the whole length, and <u>then</u> times p. You've got to add a and m first.

An indication of the consistency with which given individuals apply the same concept or procedure to items of similar type, and of the degree of association of different areas of difficulty, is presented by a set of profiles for selected individuals with respect to the notions of conjoining in algebraic addition and other notational confusions, and with respect to use of the 'rule' for simplifying algebraic expressions which requires operating with all the numerical elements first, followed by a writing in of all algebraic elements involved in the item (see Table 5.13).

Conclusions from Interviews: Elaboration of Initial Hypotheses

As a consequence of the above findings, the following hypotheses were suggested as requiring further elaboration and testing in the next phase of interviewing:

1. Error may arise as the result of children's ways of viewing letters in algebra, and that of particular importance in this regard are the following:

   a) that there is a confusion between letter as representing number and letter as representing object, the latter occurring more frequently in 'abstract' examples of the type 5x + 8y,

Table 5.13

Profile of Selected Individuals Showing Types of Error Response to Selected Items

Pupil	Item						
	Add 3 to 5y	Multiply k+2 by 3	Area a+m by p	West Ham x+y goals	Simplify 3x+8y+2x	Perimeter 2x+17	Perimeter p sides, length 5
1	Adds numbers, puts down letter	-	-	Conjoins	Adds numbers, puts down letters	Conjoins	Counts sides in diagram
2	Adds numbers, puts letter	Conjoins	-	Conjoins	Adds numbers, puts letters	-	Counts sides in diagram
3	Adds numbers, puts letter	Multiplies numbers, puts letter	Conjoins	-	Adds numbers, puts letters	-	Adds sides in diagram
4	Adds numbers, puts letter	Conjoins	Conjoins	-	Adds numbers puts letters	-	Completes diagram, adds sides
5	Adds numbers, puts letter	Multiplies numbers, puts letter	Omits brackets	(Correct)	-	Power term	Completes diagram, adds sides
6	Adds numbers, puts letter	Omits brackets	Omits brackets	-	Adds numbers, puts letters	-	p fives - verbal statement only
7	Adds numbers, puts letter	-	(Correct - brackets avoided)	Conjoins	Adds numbers, puts letters	Two sevens → 27	-
8	-	-	Conjoins	Conjoins	-	Two sevens → 27	Completes diagram, adds sides

b) that letter-as-number is frequently construed as letter-as-<u>specific</u>-number, so that the possibility of a letter assuming a range of values is not entertained, coupled with a view that 'same letter means same number', and 'different letter means different number'.

2) Error may arise as the result of alternative conceptions of the appropriate method and the need and/or ability to formalize and symbolise this method, and that this may be compounded of several aspects:

a) the use of correct but more informal 'primitive' methods which do not readily lend themselves to algebraic representation,

b) non-recognition of the formalized operational model in the arithmetic case,

c) an unwillingness to present an algebraic expression of method as an 'answer'.

3. Error may arise as the result of confusions over algebraic notation and convention, particularly with regard to:

a) conjoining in algebraic addition, and/or

b) the need to use brackets to specify order of operations.

These hypotheses were then taken as the basis upon which items for use in the second phase of interviewing were developed.

CHAPTER 6: THE ASCERTAINING EXPERIMENT - PHASE TWO

The aim of this second phase of interviewing, which took place approximately six months after the first phase, was to examine more closely the hypotheses derived from phase one.

Description of the Interview Sample

Seventeen of the children previously interviewed from two of the four schools used in phase one were interviewed again in this second phase. In addition, seven children from the fifth school selected for participation in the study, but which had been unable to take part in the earlier phase of the investigation were also interviewed. The interview sample for phase two was thus distributed among three schools as shown in Table 6.1.

The Phase Two Interviews

Design of Interview Items

Since the areas of difficulty relating to 'operating with letters' and 'convention and notation' required further investigation, sets of items designed to study these aspects were prepared (see Figures 6.2 and 6.3 which appear later). In addition, one pair of items was used in order to check on the letter-as-number/letter-as-object interpretation within the context of 'abstract' examples, as well as to affirm the 'simplification rule' of operating on numbers and letters separately which had been observed in the earlier interviews (see Figure 6.1).

One of these items, item (a), was identical to that used in the phase one interviews. In order to check for consistency of error and approach with regard to the areas under study, three other

Table 6.1

Phase Two Interview Sample

School	School Type (Location)	Number of Children Interviewed 3rd yr (Age 14)	4th yr (Age 15)	5th yr (Age 16)	Total
B	Co-ed. Comprehensive (S.W. London)	4	2	4	10
D	Co-ed.Comprehensive (Middlesex)	2	2	3	7
E	Boys Comprehensive (S.W. London)	3	3	1	7
	Total	9	7	8	24

a) Add 3 onto 5y
b) Which expression tells you what you get if you add 2 to 5a? Write down <u>every</u> answer you think is correct: 7a 7 10a 5a+2 2+5a (2+5)a None correct

Figure 6.1 Interview items - meaning attached to letters

items identical or essentially similar to items used in phase one were likewise employed in phase two.

Items for investigating the questions of conjoining in algebraic addition and the recognition of the need to use brackets in recording algebraic operations are given in Figure 6.2. Of these, items (a) and (c) were comparable with items used in phase one. Items (b) and (d) were included in order to check on the possibility that children were merely 'forgetting' to write the brackets in, and to obtain information on the kinds of expression that children saw as equivalent. It was hoped also to determine, for example, whether children who did not themselves produce a conjoined answer for algebraic addition would nevertheless select the conjoined answer as a valid possibility when presented with this as an alternative.

The issues relating to the notion of operating with letters were investigated by use of a set of items of the kind shown in Figure 6.3. The purpose of presenting items of this kind was to investigate to what extent the difficulty in handling the generalised form of a problem might disappear once it could be ascertained that a child was (a) using an appropriate method, (b) specifically attending to the operation to be used rather than the answer, and (c) aware that the 'answers' could be left in the unclosed form. In order to encourage attention to these three points, the values used in the numerical items were made relatively large, so that children would be less tempted to use more primitive 'counting-on' methods, and so that they would be less likely to leap immediately to a final numerical answer. It was decided to restrict the study in this regard to only one operation, since the CSMS Number Operations research had indicated that it was not a trivial matter for children to model word problems by an appropriate arithmetic operation (Brown, 1981a, b, c; Brown &

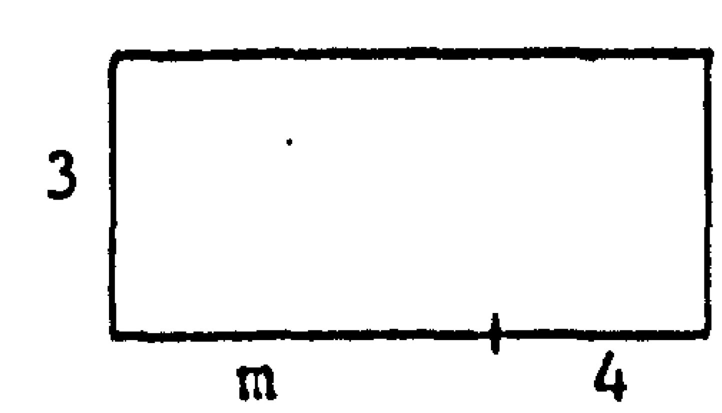
a) Multiply k+3 by 4

b) Which expression tells you what you get if you multiply e+2 by 3?

Write down every answer you think is correct:

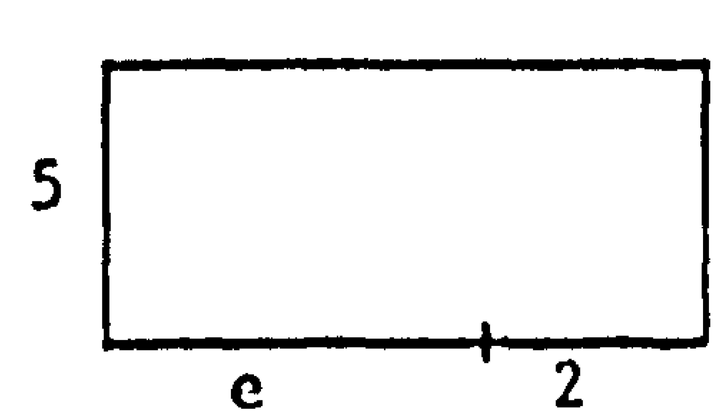
e+6
3xe+2
3x(e+2)
3e+2
3xe2
3(e+2)
e+2x3
3e+6
None correct

c) What could you write for the area of this rectangle:



d) Which expression tells you the area of this rectangle?

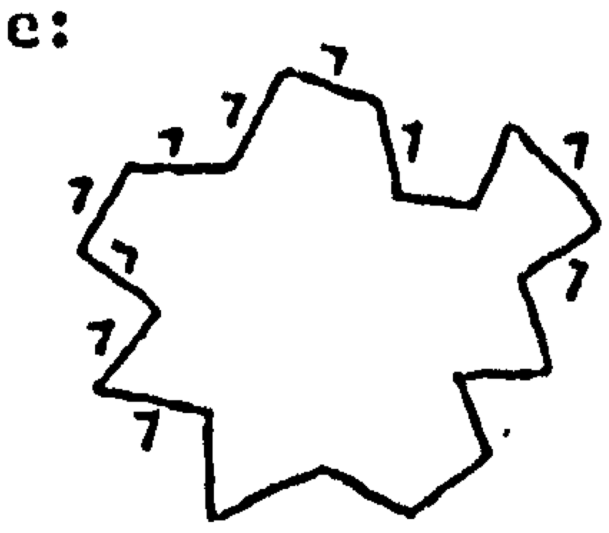
Write down every answer you think is correct:



10e
5xe+2
5x(e+2)
5xe2
5(e+2)
e+2x5
None correct

Figure 6.2 Interview items - notation and convention

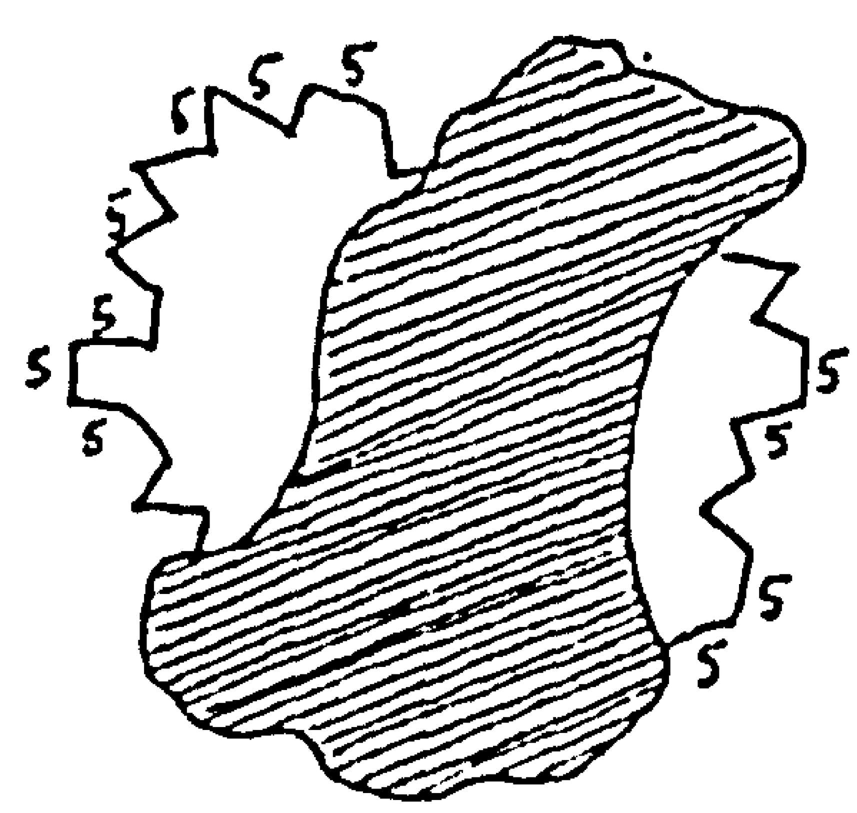
a) What could you write for the perimeter of this shape:



All the sides are of length 7.

There are 19 sides altogether.

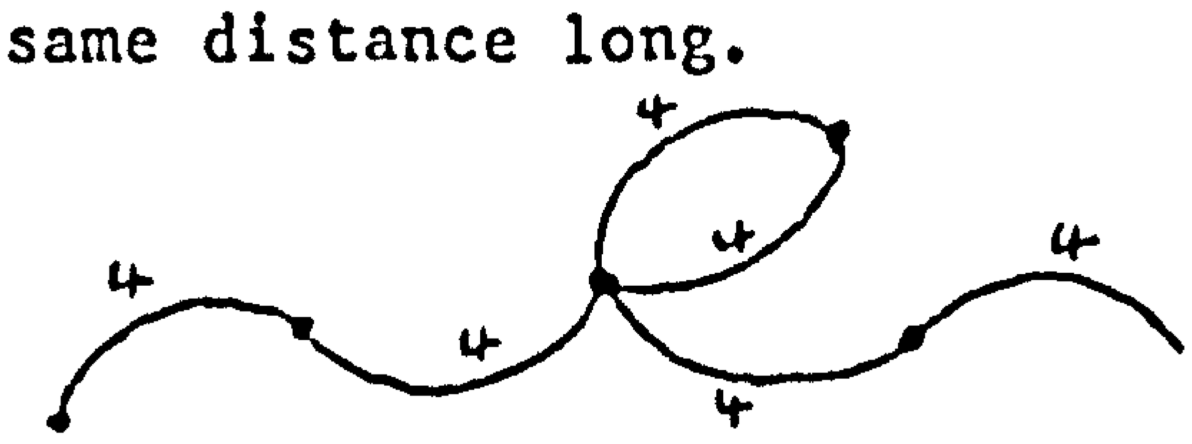
b)



Part of this shape is hidden. All the sides are of length 5

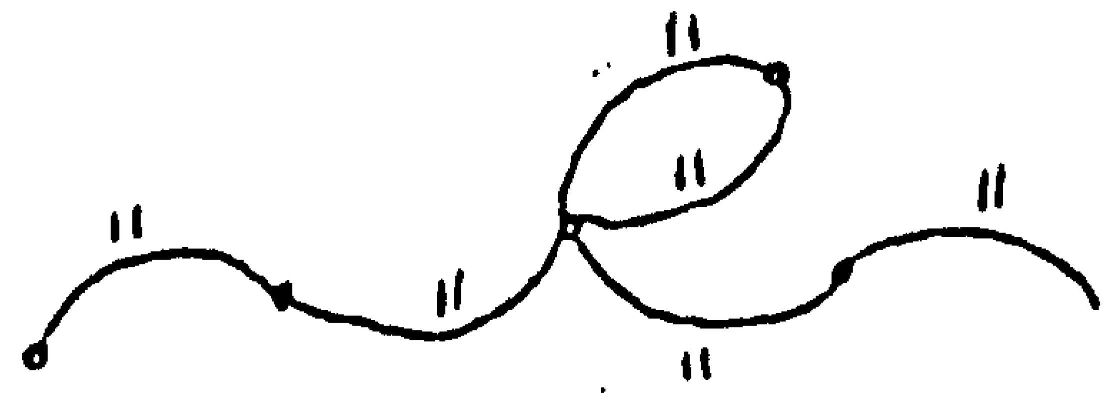
There are n sides altogether.

c) A space-ship travels in 'stages' which are all the same distance long.



If each 'stage' is 4 light-years long, what could you write for how far the space-ship goes in 97 'stages'?

d)



If each stage is 11 light-years long, what could you write for how far the space-ship goes in y 'stages'?

Figure 6.3 Interview items - operating with letters

Küchemann, 1976), and because the SESM phase one interviews had provided further evidence of this. In line with the kind of CSMS items being studied in this regard, and in view of the additional problems of recording introduced in subtraction or division problems (for example, the order in which the numbers are written), it was therefore decided to confine attention to multiplication problems. This inevitably introduces questions concerning the child's being able to handle the algebraic items merely by 'copying' the arithmetic form. However, it was felt that the interviews would clearly indicate if this was the case; also, since children had often not attempted to copy the arithmetic form in the phase one interviews, even when their attention had been directed towards a numerical instance, it was felt that this event, if it did occur, would still be of interest.

Interviews and Analysis

As in phase one, each interview lasted approximately 30 minutes and was recorded and subsequently transcribed verbatim by the experimenter. Children were again asked to record all working, and particularly the final answer. The interview transcripts were summarised as in phase one, and item analyses prepared according to the areas of interest delineated earlier.

Findings

Comparison across interviews. A comparison of the responses given by those children who were interviewed in both phase one and phase two to the common items presented in those interviews (Table 6.2) showed virtually no change in type of response. This gives a crude indication of the stability of the conceptions upon which the error patterns are based, and also suggests that the earlier

Table 6.2

Comparison of Responses in Phase One and Phase Two Interviews

Comparison Item		Number of Comparisons	'No Change'	Change to Correct	Lower Order Answer[a]	Alternative Method
<u>Phase One</u>	<u>Phase Two</u>					
Area a+m by p	Area m+4 by 3	11	9	1		1
Multiply k+2 by 3	Multiply k+3 by 4	8	7			1
West Ham scores x goals, M.U. scores y. Total scored?		11	7	1	3	
Add 3 to 5y		8	8			

a. A 'lower order answer' is one which is 'less correct' than the answer previously given (e.g. writing 'x &y' after previously writing 'x+y'), or which is incorrect after a previous correct answer.

interviewing experience did not present a significant learning situation.

Meaning attached to letters. Table 6.3 describes the proportion of children giving the indicated interpretations of the letter 'y' in the item 'add 3 to 5y', and also analyses these responses in relation to the answer given to this item.

As indicated, there appears to be some confusion over the meaning of letters in items of this kind. As before, these misconceptions are often quite explicit:

(I: Interviewer; P: Peter, 15 years)

I: And what does the 'y' mean, in a question like that (add 3 to 5y)? Does it mean anything, does it stand for anything, or is it just a letter, or what?
P: It's a letter, but it stands for something. It means 8 lots of y.
I: And what is the y?
P: Could be anything.
I: Like what?
P: Could be ..... a yacht. Could be 8 yachts.
I: OK, anything else?
P: Could be yoghurt. Or a yam.
I: Would it have to begin with y, like yoghurt, or could it be something else?
P: Think it would have to begin with y, cos you've got a letter y there. So you need a y for the start of the word.

And from another pupil, D, aged 16 years:

I: And does the 'a' mean anything there (add 2 to 5a)?
D: No, it doesn't mean anything. It's just there.

It would also seem that the level of letter interpretation may bear little relation to success in answering this type of item, in that children interpreting letters as objects may get the item right as well as wrong, while children who recognise letters are representing number may still produce the erroneous answer (see Table 6.3). It appears, therefore, that the meaning of letters may not be

Table 6.3.

Meaning Attached to Letters - Letter Interpretation

Item: Add 3 to 5 y						(N = 21)
	Interpretation of Letter					
Response	Object	Just Letter	Number	Other Interpretation	No Meaning	Not Asked
3 + 5y	2					2
8y	3	4	3	1	1	4
Value needed			1			

Table 6.4

Meaning Attached to Letters - Comparison of Items

Items: (a) Add 3 to 5y (b) Which are expressions for 'add 2 to 5a' (N = 21)					
Response (a)	(b) 5a+2 2+5a	Includes 7a, (2+5)a	Includes 7a omits (2+5)a	7a only	Not Asked
3+5y	2		1		1
8y	1	5	7	2	1
Value needed				1	

something which children take into account in answering such items; and in fact during the interviews each of 16 children who gave the erroneous answer '8y' did so by applying the 'rule' of adding the numbers and writing down the letter, regardless of the meaning which they then went on to ascribe to the letter itself.

The high degree of consistency between answers given to the two items in Figure 6.1 is shown by Table 6.4. It is interesting to note that the one pupil who answered item (b) correctly after giving an incorrect response to item (a) spontaneously noted his earlier error and subsequently corrected it. Also, two of the three pupils who selected only 7a as a correct response to item (b) explained their choice by indicating that 'the rest are sums, not answers'. This distinction between what constitutes a question or statement of method, and what constitutes an answer, appears to underlie much of children's difficulty in algebra, as will be discussed in more detail later.

<u>Notation and convention.</u> Conjoining in algebraic addition was a possibility in five of the items. Table 6.5 indicates the proportion of children producing this kind of response in each item.

Table 6.6 shows the degree of consistency between responses of this type to each pair of items described in Figure 6.2, and also indicates that a significant proportion of children who do not necessarily spontaneously produce a conjoined answer may nevertheless select one when presented with the option (five of the 20 children given both items (c) and (d) in Figure 6.2 selected the option involving 'c2' in the latter item although they had not produced a conjoined answer in the former case). As in the first set of interviews, some children were very definite in their view of the

Table 6.5

Notation and Convention (Conjoining) -

Number of Children Interviewed Giving Identified Responses

Item	Response	Number of Children	Total Interviewed
a) Multiply k+3 by 4	k+3→k3 or 3k	8	13
b) Expressions for e+2 times 3	e+2→e2	6	10
c) Area 3 by m+4	m+4→m4 or 4m	5[a]	20
d) Expressions for e+2 times 3	e+2→e2	9	16

a. Six other children showed conjoining confusion during interview.

Table 6.6

Conjoining/Use of Brackets - Comparison of Items

Item (Number of children given items = 13)		(b) Expressions for e+2 multiplied by 3: No incorrect expressions	Expressions with and, without brackets	Brackets excluded	Not asked	Conjoined answer included
(a) Multiply k+3 by 4	Correct answer		1		1	
	Brackets omitted			2	1	
	Conjoined answer		4	3	1	6

Item (Number of children given items = 20)		(d) Expressions for Area of rectangle 5 by e+2: No incorrect expressions	Expressions with and without brackets	Brackets excluded	Other wrong	Not asked	Conjoined answer included
(c) Area of Rectangle m+4 by 3	Correct answer	2					
	Brackets omitted		2	3	2	3	5
	Conjoined answer		3		2		5
	Other wrong	1				2	

meaning of the conjoined term:

(I: Interviewer; C: Christopher, 15 years)

C: (In reply to item (d), Figure 6.2) 5xe2 and 5(e+2) - that's all.
I: Why do you say 5xe2?
C: 5 times e2. So that means the e plus 2, the e and 2 put together, times 5. That's what the answer should be, 5 times e2.
I: What does e2 mean?
C: The answer to e add 2 ..... 5 times e2 ..... The e2 means you have to add the e and 2 before. So it's .....I think I'll stick to that one (5xe2) actually, because you've got to add the e and 2 first.
I: And e2 tells you do do that?
C: Yes, to get the answer.

Other children showed evidence of viewing the conjoined term not only as the result of having performed an addition between two numbers as in the above example, but also as reflecting an 'implicit' addition in the place-value sense:

(I: Interviewer; W: Wayne, 15 years)

W: (Explaining the meaning of 'y' in 'add 3 to 5y'.) y could be a number, it could be a 4 making that (5y) 54. Or it could be a 5 to the power 4, making it 20 (writes 54, 5^4).
I: How could you know which one it was, out of those two?
W: (Pause) I don't know!
I: Do you think it could be either, or do you think it's one of those only you're not sure which?
W: No, it could be either, you can't really say.
I: So it's either ..... 5 with a little 4, or it's ..... read the other one out.
W: What ..... five four ..... no, fifty four!
I: Fifty-four?
W: Yes.
I: Could y be anything else, besides 4?
W: Yes, 7, 8 anything!
I: So y could be any number? (WA nods). Suppose I made it 23. What would you write down then?
W: Oh! (laughs). Well! (laughs) five hundred and twenty three! But I dunno - it doesn't sound very promising! I dunno. Wait, it could be 28, 5 plus 23 ..... yes ..... (pause). There again it could be 5 to the power of 23. (Writes 5 + 23, 5^{23}).

The confusion between products and powers has already been noted in Chapter 5 and continued to be observed in these later interviews.

Table 6.6 also reveals a high consistency between each pair of items (a), (b) and (c), (d) (Figure 6.2) in terms of use and recognition of need for use of brackets in algebraic expressions involving two operations. As indicated in the tables, very few children used or selected bracket-expressions correctly, with two out of 20 and three out of 16 providing correct answers to the 'area' items (c) and (d) respectively, and one out of 13 and none out of 10 doing so for the 'multiplication' items (a) and (b). A number of children considered expressions with or without brackets to be equivalent, maintaining that 'you can put the brackets in if you want, it's just the same' and a similar number excluded the bracket expressions altogether, either because they considered their presence unnecessary, or more rarely, because they did not know what the brackets meant:

(I: Interviewer; M: Marie, 16 years)

M: (Having selected the expressions 5xe+2, e+2x5, 5x(e+2) and 5(e+2) in response to item (d), Figure 6.2.) ..... only I'm not sure about those two (expressions with brackets). I don't think they're right (crosses them out).
I: Why don't you think they're right?
M: It's the brackets. I don't know why it's there. It might be alright. Only I don't think so.
I: But you think 5xe+2 and e+2x5 are right?
M: Yes. It (5xe+2) says multiply this side (5) by this side (e+2). And so does this one (e+2x5).

In each case in which the expressions 5xe+2 and e+2x5 were selected as correct, justification was given as in the above example, by reference to the context (area of a rectangle) of the particular item. Evidence obtained subsequently (see Chapter 7) suggested that where a particular context is not defined, such expressions tend to be evaluated from left to right, so that the two expressions may then be regarded as non-equivalent. Also of interest was the number of children who did not recognise the equivalence of the two bracket expressions (two out of the five children selecting bracket statements in item (b), and seven out of the 12 children selecting bracket statements in item (d)). However, all the children given this item interpreted the item appropriately and were able to explain correctly the method required to find the area. In this instance, therefore, while children were able to describe a correct and appropriate method (and it seems that finding the area of a rectangle is one formal method which these children did tend to know), they were unable to correctly record the answer, partly because of various notational confusions. However, there was evidence that these confusions were not solely related to the question of algebraic representation per se, but also may reflect an insecure understanding of representations in arithmetic. This point will be discussed more fully in the next chapter.

Operating with letters. All the pupils interviewed in phase two were presented with a series of numerical items of the kind shown in Figure 6.3 (items (a) and (b)). In each case, the pupils were asked what they could write for the required perimeter etc., and told that they could 'leave the answer in its un-worked-out form'. Most of the pupils interviewed adopted the multiplicative operation for these items sooner or later, even if they began by suggesting a repeated addition approach (although three pupils in the 'space-ship' item (item (c) in Figure 6.3) recorded a division problem, showing themselves during interview to have been mis-cued by the phrase: 'What could you write for how far the space-ship goes in 97 stages', apparently confusing this with 'goes into', which they normally associated with division). One pupil, however, consistently used repeated addition even though the language he used appeared to reflect a mutliplicative approach:

(I: Interviewer; A: Adam, 13 years)

A: (Long pause) 46 (for the perimeter of the shape with 23 sides all of length 2)
I: How did you get that?
A: Twenty-three 2's.
I: Right, could you write that down? Show me how you did it?
A: (Starts to write a vertical column of 2's).
I: Oh, I see, and is that how you did it just now, when you did it in your head?
A: I did the times table.
I: How?
A: I went 2,4,6,8,10 ......
I: Oh, I see you counted on like that. How did you know when to stop?
A: When I got to 23.
I: I see. But you write it down like that (the column of 2's)?
A: Yes.
I: Is there a quicker way of doing it, without writing down all those 2's?
A: (Pause). Yes. You can do two 23's (writes down two 23's in a column).
I: Suppose I had nineteen sides and they were all 7 units long?
A: (Pause) You'd do seven 19's (writes 19 seven times in a column)
I: Is there a quicker way of writing down seven 19's than that?
A: No, only doing all the sevens, but that's longer.

Faced with the problem of the perimeter of the shape with n sides all of length 5, this pupil (A, above) used an alphabetic substitution (n=14), and added the 14 five times, as did another pupil (M, see below) who reverted at this stage from multiplication to the more primitive adding-sides approach. However, it was clear that this adopting of an alphabetic substitution procedure was more the outcome of the use of an inappropriate method and of the desire to make sense of an otherwise meaningless situation, rather than the expression of a deeply-held belief that the values represented by letters did in fact relate to their ordinal position in the alphabetic sequence:

(I: Interviewer; M, Michelle, 14 years)

I: Can you write down anything for the perimeter?
M: No, because n probably stands for a number and if that's n, I can't get how many sides that is. And I've got to add up all the sides, all the 2's, so I need to know how many. Unless n stands for ..... like say n's in the alphabet, somewhere along the alphabet, so going along the numbers ..... if n stands for one of those, then you can say what n is.
I: How do you mean?
M: Well, say n's 14.
I: How did you get that?
M: Same as I said before, to get n for a number, I got 14. It's 14 along.
I: Oh, did you count along the alphabet?
M: Yes, so that's 14, so I took that for n, so it should be ..... 28, to make up the quantity, the perimeter.
I: 28?
M: Yes, you add up all the 2's.
I: Could n be another number, or does it <u>have</u> to be 14?
M: Well it could any n number, but if you write the question out like that, without <u>any</u> indication of the number, then there's nothing ..... The only thing you can do is go along the alphabet and take whatever number n comes under to tell you how many two's you need.

The majority of the pupils, however, were successful in recording their answers to the numerical items as multiplication. When presented with the algebraic items (those pupils who had previously received 'perimeter' type questions received ths 'space-ship' algebraic item first and vice versa) a large proportion of these pupils were now able to produce a correct algebraic statement (see Table 6.7), and to explain their answers, for example:

(I: Interviewer; M: Mark, 14 years)

M: (In reply to the 'space ship' item (d) in Figure 6.3.) Well, it'll be 11 times y.

Table 6.7.

Operating with Letters: Performance on Algebraic Items Following Trial on Arithmetic Items

Item[a]		Algebraic Items (Perimeter, Spaceship) Correct, horizontal recording	Correct, vertical recording	Alphabetic substitution	Other numeric answer
Numerical Items (Spaceship, Perimeter)	Correct, horizontal recording	16		1	
	Correct, vertical recording	2	2		2
	Repeated addition			1	

a. See Figure 6.3 for examples of items used.

> Each stage is 11 light-years long and you've got y number of stages, so you times 11 by y.

(I: Interviewer; T: Tristan, 16 years)

T: What'd it be, for y stages, well, it'd be 11 times y (writes 11 x y).
I: And why did you write 11 times y?
T: Well, 11 light-years, times so many stages, would give the total distance.

and for the perimeter of n sides each of length 5:

(J: Janet, 16 years)

J: Well, n times 5. There are n sides, each of them are 5, so to find the answer you times n by 5.

Where the method required was clear to the pupil and could be explicitly stated, the presence of the letter rather than a numeral appeared to cause relatively little difficulty. It is suggested that the use of the numerical 'training' examples, by focussing the child's attention on the operation to be used rather than the answer, and by 'allowing' the children to leave their answers in unclosed form, thereby legitimising the notion of undetermined answers, served to reduce much of the difficulty which children otherwise have with generalised problems of this kind. Certainly the proportion of children successfully answering the generalised 'perimeter' problem was far higher under these circumstances than it had been in the first interview phase (see Table 6.8).

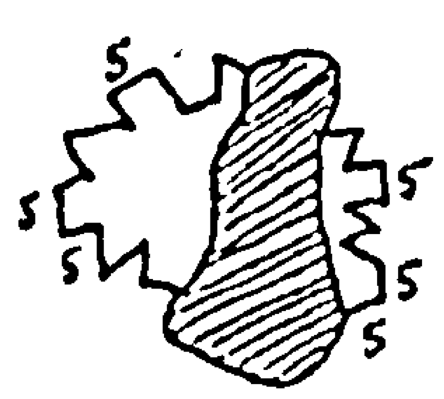
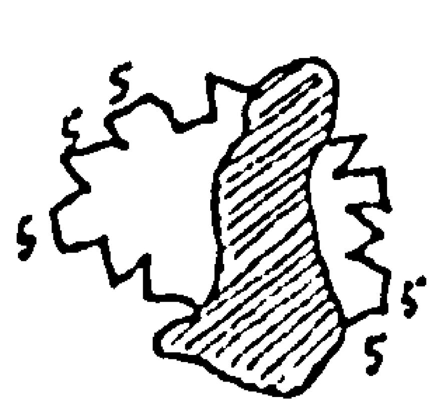
That children appear to require (or consider that the question or the teacher requires) a particular numerical answer was still apparent even under these conditions, however, suggesting that even though the correct algebraic expression is supplied, children may not view it as a 'proper' answer:

(I: Interviewer; A: Andrew, 15 years)

A: If it travels in one stage 11 light-years then in y stages you don't know how much y

Table 6.8.

Proportion of Children Interviewed Giving
Correct Answer to Generalised Perimeter Item

Item		
CSMS Pretest	(Number of children given item = 55)	
Perimeter of:  (n sides)	Answer correct	0.11
Phase One Interview	(Number of children given item = 13)	
Perimeter of:  (p sides)	Answer correct (horizontal or vertical recording)	0.15
Phase Two Interview	(Number of children given item = 24)	
Perimeter of:  (n sides)	Answer correct (horizontal or vertical recording)	0.84

stands for, so if I did, you'd multiply, so I'd do 11 multiplied by y, but I can't get the answer if I don't know what y is (writes 11 x y = 11y).

(C: Christopher, 15 years)

C: How long is y stages? You really have to know what y is. As well as the length. You'll have to make it up.
I: So what would you do?
C: Well, then ..... say it was 15 stages, then you'd do 15 times 11 (writes 15 x 11).
I: OK, so that would be it if y was 15, now can write something if I don't know it's 15?
C: Only y (writes y x 11).

and from 16 year old Marie in reply to the item concerning the perimeter of a shape with n sides each of length 5:

(M: Marie, 16 years)

M: n times 5. It's the number of sides times how long each side is, only you don't know how many sides, so all you can do is n times 5.
I: So the answer is n times 5?
M: Well, you can't give a proper answer, because you don't know what n is. If I knew n, I could work it out, but as it is all you can put is 5n.

That some at least of the observed difficulty in algebra may relate to a difference of opinion between teacher and child as to what constitutes a legitimate answer in algebra, and indeed what constitutes algebraic activity, is perhaps most clearly shown by the following discussion on the 'space ship' item:

(I: Interviewer; W: Wendy, 14 years)

W: There's a letter there.
I: What does the letter mean?
W: It's telling you how many stages
I: Right. Can you write anything for how far the spaceship goes?
W: Times it by that number.
I: OK, could you write something down, then, to say how far it goes?
W: What, shall I write down what I would do? (Writes 'if y was a number I would times it

by 11'.)
I: Now can you write that without using words, using maths instead?
W: What, how'd you mean, like 11 times y?
I: Yes, OK.
W: (Writes '11 x y'). Is that it?
I: Fine.
W: What, is that all it was? Why didn't you say so, I thought you wanted the answer.
I: Do you mean a particular number?
W: Yes!
I: Well, is there a particular number answer?
W: No! Not unless you know what y is.
I: Well, then, how could you give me a particular number answer!
W: Well you can't, but I didn't know you only had to put that.

For many children it seems that mathematics is an empirical subject which requires the obtaining of specific numerical answers. Consequently, children may be reluctant to give an algebraic answer, assuming that something else must have been intended, or they will derive a correct algebraic statement but then use various substitution devices in order to obtain a numerical answer. In Wendy's case (W, above), she had previously been unable (or unwilling) to write anything at all in answer to generalised problems, as can be seen from her response to the 'West Ham' item presented earlier in the interview:

W: (West Ham scores x goals and Manchester United scores y goals: What can you write for the number of goals scored altogether?) Don't know. It's the x and y, I don't know how many goals they are.
I: Right, if you did know how many goals they are, how would you find the number of goals scored altogether?
W: By adding them up.
I: So can you add those?
W: No, because they're not numbers.
I: Could you write down what you'd do?
W: What, adding them up? Well, what am I adding up!
I: Suppose West Ham scored 2, and Manchester United scored 5 .....
W: 7 goals. (writes '7 goals').
I: And can you write down what you do to get 7?
W: (Laughs) Just add them up! You don't

write it down.
I: OK, let's say West Ham scored 73 goals and Manchester United scored 96 .....
W: Well, you add them (writes $\begin{array}{r} 73 \\ 96\,+ \\ \hline 169 \end{array}$).
I: Can you write anything down with the x and y, to show you're adding them?
W: No! (Laughs).

Here it is suggested that Wendy's view of the requirement for a specific numerical answer, together with a tendency to disregard the arithmetic representation of the problem so that she does not have an explicit arithmetical model from which to generalise, is largely responsible for her lack of success in handling algebra at this level.

Child methods. The lack of an explicit model in arithmetic appeared to characterise many interviewees' behaviour. While children may have successful ways of solving easy problems in mathematics, it seems that these methods are often not explicitly attended to, so that they are not made the object of reflective scrutiny nor are their limitations explored. One consequence of this is, of course, that they are not easy to symbolise in any concise mathematical form. This appears to be due to the twofold issue of having an appropriate method in the first place, and of being explicitly aware of that method as an entity in its own right. As a result of the investigation into children's methods in other areas of mathematics as well as algebra, it is suggested that these 'child-methods' may be described as being:

1. intuitive, i.e. based upon instinctive knowledge : not systematically reflected upon and not checked for consistency within a general framework.
2. primitive, i.e.tied closely to early experiences in mathematics.
3. context-bound, i.e. elicited by the features of the particular problem.
4. indicative of little or no formal symbolised method.
5. based largely upon the operations of counting, adding and combining.
6. worked almost entirely within the system of whole numbers

(and halves).

These methods are also characterised by being strongly adhered to and reluctantly abandoned by the child, possibly because of previous successful usage. Indeed, it might be suggested that, because children have been successful in their use of such methods (with easy problems), they see no point in attempting to master the more formal methods taught in the classroom. For example, the finding of perimeter by 'counting on' the sides, or the finding of rectangular area by counting the unit squares contained within the figure, are suggested examples of such informal methods. Other examples include the solution of division problems by grouping or successive subtraction (see also chapter 7), and the solving of ratio problems by successive addition of parts (e.g. Hart, 1981a,c, 1983). In each case the method can be successfully applied in the case of 'easy' problems but may be inappropriate, or applied only with great difficulty, when the problem becomes more complex (see Booth and Hart, 1983; Hart and Booth, 1981). It is suggested that much of the difficulty which children have in algebra, and in mathematics in general, may be related to children's general 'informal strategy' approach to mathematics, and the attempted use of these non-explicit and informal child-methods to situations in which they are inappropriate.

This issue relating to the kind of method used and the formalization of method can be seen to be distinct from (though necessarily related to) the question of symbolisation. Even where the child does have an appropriate method which is formalized at least verbally, it is not necessarily a trivial matter for the child to symbolise this in acceptable or appropriate mathematical terms, as has been shown by the findings on conjoining in algebraic addition and on the use of brackets, and is further illustrated by the following:

(I: Interviewer; S: Stephen, 14 years)

S: Add 3 onto 5y, is 8y (writes '8y').
I: Can you write the question down as well, but using maths?
S: (Pause) (Writes '+3 — 5y'.) Add 3 onto 5y.

The way in which children record operations in arithmetic may also have relevance for ease of transfer to the algebraic case. For example, some children appear to have a preference for recording arithmetic operations in vertical 'column' form (see, for instance, the interview with Wendy, described earlier). Of the 24 pupils interviewed on the numerical perimeter/space-ship type items (Figure 6.3), six did in fact use this form of recording. Whilst appropriate and convenient for computational activity, this form of recording is not usual in algebra, and may well 'encourage' children to seek a 'closed answer'. Table 6.7 also shows the response pattern across the numerical and algebraic items from Figure 6.3 for the six pupils who used a column or vertical form of recording in the arithmetic case, and also for the pupil who consistently used repeated addition. Two of these pupils also used the vertical column format for the algebraic items by making a numerical substitution and recording a numerical answer. These three pupils were, in fact, the only ones who did not produce a correct algebraic answer to these items.

Conclusions from Interviews : The Basis for Development of the Teaching Module

The data from the two interview phases suggest particular areas of difficulty in algebra:

1. Children's interpretation of letters:
   a) Children may handle letters as 'objects', especially in more abstract examples.
   b) Children may interpret letters as specific unknown numbers in situations which require consideration of them as generalised number.

2. Non-formalization/non-symbolisation of method:
   a) Children may use informal or more 'primitive' methods which are difficult to symbolise concisely.
   b) Children may not have an explicit model in the arithmetic case from which to generalise (even where the 'method' is appropriate).
   c) Children may use a form of recording in arithmetic which is not appropriate to algebraic recording.
   d) Even where children use a formal method and symbolise it correctly and appropriately, they may not see that this is an appropriate thing to do.

3. Understanding of convention and notation:
   a) Children may attempt to perform an algebraic sum giving a conjoined term as answer ($a+b \rightarrow ab$). This may be a consequence of a viewpoint which does not recognise an expression such as $a + b$ as a legitimate 'answer', but rather as indicating an operation which still needs to be completed.
   b) Children may see no need for brackets and consequently not use them. This may be partly a result of a viewpoint by which expressions are interpreted within a given context which defines the required order of operations.

The next phase of the research was the teaching experiment phase, in which the difficulties indicated above were used as a basis for developing a 'teaching module' designed to help children overcome these particular difficulties. This teaching programme was prepared for exploratory investigation in a set of small-scale teaching experiments, each involving four or five children.

CHAPTER 7: TEACHING EXPERIMENTS

Having ascertained various areas of difficulty in algebra which appeared to contribute towards the errors children made in this topic, the next phase of the research involved an investigation of how these difficulties might be alleviated by instructional intervention. Since many of the problems identified appeared to relate to and interact with each other (and indeed this was supported by analyses of the association between different areas of difficulty, for example see Table 5.15), it was decided that in order to be effective, the instructional sequence should address all these areas of difficulty. The sequence was developed and tested through a series of small scale teaching experiments by which the researcher monitored as closely as possible the interaction of a small number of students with each step of the proposed instructional sequence. The aim of these small scale teaching experiments was twofold: 1) to gain further evidence concerning the degree of consolidation of the areas of difficulty revealed by the interviews, and 2) to monitor any changes in cognition which the instructional sequence brought about. The first could be achieved by observing the incidence of the areas of difficulty in a (partially) new group of students, and by noting the ease or lack thereof with which various misconceptions or erroneous viewpoints were abandoned. The purpose in monitoring changes in cognition was to assess the effectiveness of the teaching sequence in helping children to restructure their thinking and so avoid making the errors. It was also intended that the instructional programme developed would be of practical use in suggesting ways of teaching elementary algebra in school which might help avoid the development of the misconceptions described in the previous chapters, or help correct them if already formed. For this reason, it was deemed necessary that the teaching sequence should be suitable for use with whole classes (since this style of teaching appears to be most prevalent at secondary level), and that it should not, at least initially, be of too long duration. The sequence was therefore designed to cover a period of approximately six 35-minute lessons, and was designed in a form which would be appropriate for whole-class use. A requirement of the teaching experiment paradigm adopted for the research is that the instructional programme developed in the individual or small group situation be verified with larger groups (see Chapter 3). The organisational arrangements existing in many schools require that 'large groups' of children be whole classes: certainly this was the case in the schools used in the present study. In addition, since the teaching programme was intended to be of ultimate use in the class situation, or at least form the basis for curriculum development within that context, it was desirable to monitor the programme's effectiveness in this domain. Consequently, having been developed in the small-group teaching experiments (the subject of this chapter), the teaching sequence was then tried out with larger groups (half-classes) as a prelude to evaluation with whole classes taught by regular teachers, as described in Chapter 8.

This use of whole classes for the verification process is in one sense less appropriate, since the teaching programme in question had been designed specifically to address particular misconceptions which only some of the children in a given class would be likely to have. The more appropriate 'large group', from the verification point of view, would therefore have been a larger group of children all identified as making the errors under study. However, it was thought useful to ascertain that the programme devised would not, in fact, be disadvantageous to children who were not making the errors, especially if an ultimate view of the programme was to be as one which could be used in a whole class setting. For this reason, and for the reasons of school requirements indicated above, it was considered appropriate to use whole classes for this phase of the research in the present case. The decision to use the regular class teacher as the instructor in this part of the study was also made on the grounds of providing a more realistic test of the programme's feasibility in the normal teaching situation.

The present chapter also describes the results of a paper-and-pencil test administered separately at this time, and which aimed to gain more information on the prevalence of the notions concerning conjoining in algebraic addition and non-use of brackets which the interviews had revealed. Illustrations relevant to the latter notion which were derived from interviews with mathematically-able students as well as from discussions with children of average mathematical attainment are also presented.

The Teaching Sequence

General Principles

It was decided that each area of difficulty described in Chapters 5 and 6 should be addressed in the following manner:

	Area of difficulty	Teaching requirement
1.	Children's interpretation of letters as objects or as specific unknown number	Introduction of idea of letters representing generalised number, i.e. a letter may stand for a range of values, not just one specific value.
2.	Formalization and symbolisation of method	Concentration on attention to the formal structure of problems and on the need to

		represent the 'rule' for a problem solution correctly and unambiguously.
3.	Notation and convention: Conjoining in algebraic addition	Consideration of the legitimacy and status of indeterminate answers, and the meanings which might be ascribed to them. Direct comparison of sums and products and re-iteration of the meaning of the conjoined term.
4.	Notation and convention: Non-use of brackets	Consideration of the possible ambiguity of expressions written without brackets, as a prelude to introducing the need for the use of brackets. (This had to be modified as a result of the preliminary teaching trials, as will be discussed later.)

While there are probably many ways in which a teaching sequence could be designed in order to meet these requirements, one context seemed particularly suitable for this purpose, namely that of giving instructions to a machine to solve given problems. One might think of this as analogous to programming a computer. The teaching programme was therefore designed around an idealised 'mathematics machine' (Figure 7.1), for which all instructions could be written using simply the 'language' of mathematics.

The use of such a model enables the teaching requirements indicated previously to be handled by focussing attention on the need to make explicit the procedure by which a problem is to be solved, and on the need for precision in representing that procedure. It also permits the introduction of letters as generalised number or variable by using letters for number locations (as does a computer). The teaching programme was also designed to address the problem of indeterminate answers by specifically considering the kind of answer which the 'mathematics machine' can produce in response to an

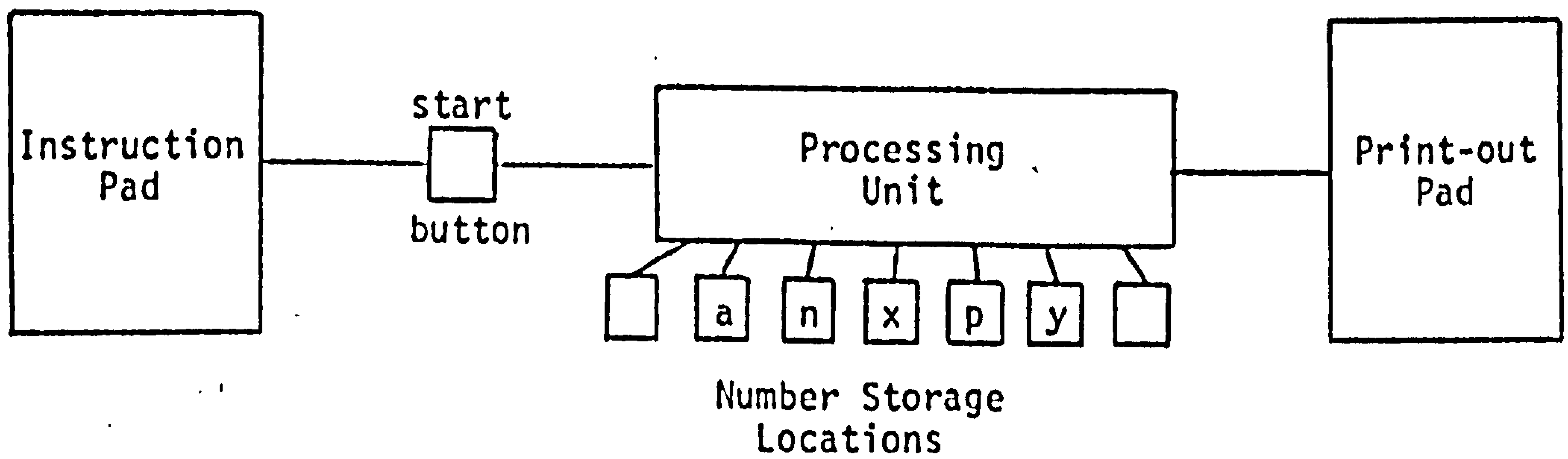


Figure 7.1 'Mathematics machine' model

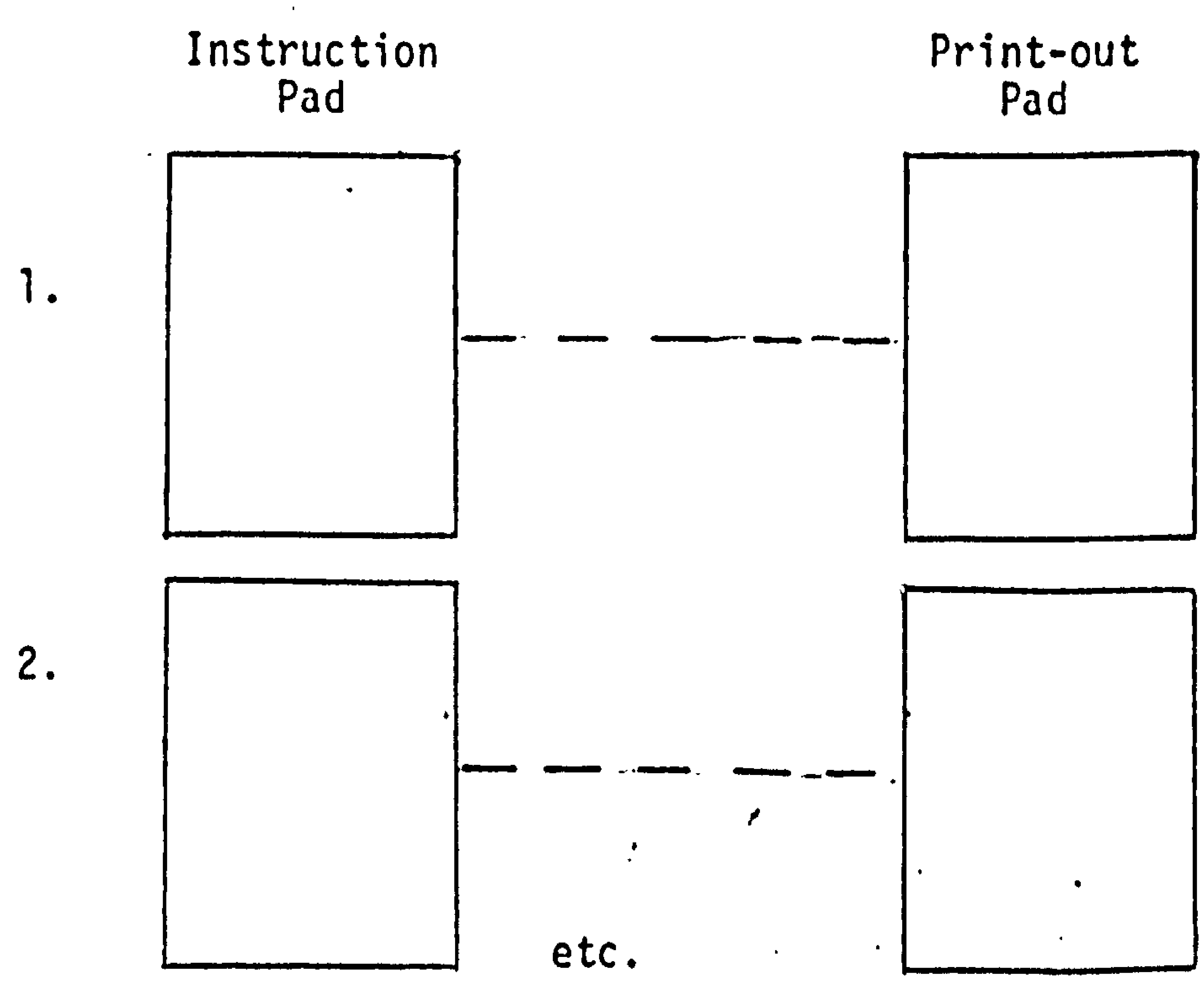


Figure 7.2 Answer blank for children's use in recording work done in teaching programme. Each sheet had room for 15 questions.

instruction such as 'add 3 to any number' (or 'n+3') when no replacement value for n is given. All operations were (initially) required to be written in full (e.g. nx3), and only towards the end of the sequence was the notation '3n' introduced as an abbreviation for the product nx3 or 3xn.

It was also felt that the use of a 'machine' context such as this would serve one other important function. One aspect which had seemed during the interviews to be reflected in each problem area was that relating to the child's perceived need for or purpose of using letters or writing general statements. Children often saw no point in using letters. It was considered that children would not take the use of letters seriously until a rationale for their use became apparent to them. It was anticipated that the introduction of the 'mathematics machine' context would go some way towards fulfilling this objective.

Outline of Programme

The teaching programme as originally proposed was divided into six components. (A sample of each worksheet used is given in Appendix 3. The full teaching sequence, together with teacher's notes, appears in Appendix 5, but for purposes of simplification only the final version is given there.) Throughout the programme the child's task was to write the instructions (using mathematical notation) which would enable the machine to solve each problem or class of problems. In each instance the answer or 'print-out' was also recorded. This was done in order to enable the child to appreciate that specific numerical answers are not always appropriate. Where numerical answers were appropriate the children performed the machine 'processing unit' function by computing answers using a calculator. Instructions and print-outs were recorded on printed 'answer-blanks' (see Figure 7.2).

The six components or 'steps' in the programme were as follows:

Step — **Outline of Content**

1. **Introduction**

   Description of 'mathematics machine'; discussion of rules for programming the machine - at this stage brackets are introduced as being needed by the machine in order to 'know' which of two or more operations to perform first.
   Example: 'I want the machine to add 15 and 8. How will I write the instructions?' For each example, all alternative forms of recording are discussed, and attention drawn to the fact that not all operations are commutative. A calculator is used as the 'processing unit' and instructions and answers are recorded on the answer-blanks provided (Figure 7.2).

   Note: In this preliminary version, only whole numbers were used. The final version was amended to include decimals.

2. **Number Operations**

   Here the examples are simple word problems and the child's task is to write the instructions for solving each problem, giving all alternative forms where appropriate. As before, the 'print-out' answers are derived by calculator.
   Example: 'A fat man wants to lose 57 lbs weight. So far he's lost 29 lbs. How many more lbs. must he lose?'

3. **Generalisation I**

   The notion of the use of letters to represent a range of values is introduced as is the idea of the indeterminate answer:
   Example: 'I want the machine to add 5 to any number I give it. How will I write the instructions?' Again all possible alternative forms are discussed.
   Example:'Complete the print-out':

Instruction	Print-Out
a+6	
a=4	----
a=110	----
a= ?	----

The idea of specific replacement values is introduced and the numerical 'answer' for each given replacement value is calculated. This situation is contrasted with the case where no replacement value is given: 'What kind of answer will the machine print out?'

Note: Unforeseen confusions over the meaning of terms such as 'any number' and 'same or different values' resulted in an elaboration of this section in the final version .

Example: 'I want the machine to add any two numbers I give it.' Discussion of when the same letter is needed, and when different letters are required. Introduction of idea that:
same letters -- same value
different letters -- same value OR different values.

4. Generalisation II

Simple problems.
Example: 'Find the area of any square'. Discussion of the difference between a x a and a x b, etc.

5. Notation

Note: Symbolic representation of this kind needs considerable reinforcement as children do not readily attach meaning to such expressions. This section was consequently extended in the final programme.

Discussion of the equivalence of expressions such as a+a+a and 3 x a.
Example: 'Find the perimeter of 6-sided shape which has all sides the same length.'
Representation of 3 x a by 3a and contrast with 3 + a.
Example: 'What do these expressions mean? Write your answer as many ways as you can.'
4m; 6+y; etc.

6. Consolidation

Example: Discrimination exercises. 'Tick the correct statements.'

a) Divide 8 by any number I give	x - 8 8 - x
b) add together any three numbers I give	x+x+x a+b+c 5+x+y

The Teaching Experiments

Sample

Three groups of children were selected comprising one group of six 13-year-olds and two groups of five 14-year-olds from two schools, one of which was new to the study (see Table 7.1). The children selected were chosen on the basis of their responses to the CSMS Algebra test according to the criteria used for the selection of interviewees and described in Chapter 5.

Design

Pretest. A subset of the CSMS Algebra test items comprising the 'error' items chosen for study by SESM (reported in Table 4.6), together with selected other items suggested to be relevant to the areas of difficulty identified through the interviews, was used to give a measure of pre-treatment performance. (The pretest items are given in Appendix 6.)

The instructional phase. The teaching material was presented by the researcher in sequence and each section and worksheet was discussed with the children. All comments and questions raised by the children were noted, selected sessions were tape recorded, and all worksheets were collected to obtain a written record of answers (i.e. the completed 'answer-blanks', see Figure 7.2). For each of the three groups, the teaching programme lasted the equivalent of six 35-minute lessons and took place over a period of five to six consecutive days.

The immediate posttest. A posttest comprising items parallel to the set of CSMS items used to measure pre-treatment performance was prepared (Appendix 7), and administered on the day following the final teaching session.

Table 7.1

Small Scale Teaching Experiment Sample

School	School Type (Location)	Year Group	Class Description	Number of Children
C	Co-ed. Comprehensive (S.W. London)	3rd Yr (Age 14)	Middle ability band (CSE)	5
F	Co-ed. Comprehensive (East Sussex)	2nd Yr (Age 13)	Mixed ability class	6
		3rd Yr (Age 14)	Middle ability band (CSE)	5

The delayed posttest. The pretest set of CSMS items was compiled into a separate test for use as delayed posttest , and this was administered four months after the immediate posttest . This four month interval included six weeks' summer vacation. Teachers were asked to provide details of any algebra teaching done by them in the intervening period. Teachers were not at this stage given any detailed information concerning the nature of the experimental teaching module nor of the errors under investigation; consequently any teaching of algebra done by them would have been presumably carried out according to their usual style.

Results of the Teaching Experiments: General Observations

All three groups of children seemed interested in the general approach taken by the teaching programme though doubtless this may have been prompted by the 'computer' connotations. In both schools, several children from other classes who were not involved in the study came to ask if they were going to do the same work. It would seem, therefore, that the programme was fairly motivating.

Motivation was also quite likely to have been enhanced by the use of calculators during part of the sequence. In both schools, children below fourth-year level (age 15) were not at that time 'allowed' to use calculators in mathematics. It was interesting to note, however, that all the children involved in the teaching experiment did in fact carry calculators with them.

The use of calculators had a second benefit. None of the children were accurate in their recording of number operations, especially those involving subtraction or division. They were also totally unaware of the mistakes that they were making. A discussion of the kind of answer that was expected from a consideration of the problem, followed by the use of calculators to compute the value of the expressions which they had actually written, was extremely useful in bringing these errors to the children's attention, and in encouraging them to think more carefully about such expressions and the way in which they should be recorded.

At the start of the programme, the general attitude towards algebra was fairly negative. All the children said that they could not do it, and did not really see the point in learning it:

> I hate it (WM, 14 years); don't understand it (GM, 14 years); what's it for? (CS, 14 years); nobody needs to know algebra (RD, 14 years); letters are stupid, they don't <u>mean</u> anything (CB, 13 years); I can't do it anyway (MM, 13 years).

The idea of using letters to write general 'rules' to enable the machine to solve whole classes of problems seemed to make sense to the children, although several appeared to have some difficulty in fully grasping the idea, at least initially. It appears that the introduction of algebra requires two problems to be addressed, namely that of justification of use or purpose and that of conceptual difficulty. It is suggested that children who are not persuaded on the former point will make little effort to try and come to terms with the latter. Certainly the evidence from the teaching experiments clearly indicated this to be the case. By presenting a situation in which the use of letters and general statements was perceived by the children to be both reasonable and useful, it is suggested that the teaching programme was effective in addressing the former issue.

Also of evident value was the presentation of a schematized 'model' of the machine which gave the children a more concrete picture of the situation and in particular presented a breakdown of the procedural steps involved, from the giving of instructions through the processing or computing stage to the representation of an output or answer. It is suggested that this helped to focus the children's attention on a form of mathematical procedure with which they are not often explicitly aware. During classwork, it was interesting to note that several children drew in the 'start' button on their answer sheets, and actually pressed it after recording the instructions to each problem and before recording the appropriate output. The writing of both instructions and output was also of value in discussing ways in which expressions could be read or interpreted. For example, 'n+3' can be read as 'add 3 to n', 'the number which is 3 more (or bigger) than n', 'the answer you get when you add 3 to n', and so on. The children's level of literacy in this regard was extremely low at the start of the programme, and they were evidently surprised that there were alternative ways of viewing the expression. In particular, the fact that an 'answer' can look the same as an instruction or 'question' appeared to be a revelation to them.

Also of apparent sense to the children was the idea that the machine had to be instructed as to which operation out of two or more had to be performed first. This idea was readily accepted by the children and brackets were consequently used by them without apparent difficulty. Indeed, they appeared to take quite a paternalistic attitude towards the machine in this regard, and even reprimanded the experimenter-teacher when she accidentally omitted them in class demonstration examples. However, it was not clear that the children accepted that brackets might also be needed in 'non-machine' mathematics, and the teaching-experiment test results (discussed in the next section) indicate that this aspect of the algebra being 'context-bound' was a weakness of the programme.

As a final comment on the interest which the programme seemed to arouse, three of the children participating in the programme came to see the experimenter during a later visit to the schools to say that they had bought computers and were using them at home. This may indicate, of course, that the programme might be less motivating for children who are already familiar with computers and their use (however, it may be that such experience may result in children not making the errors under study in the first place, so that such a programme would be less appropriate for them in any case).

Results of the Testing

Changes in total score. Figure 7.3 shows the pre- and posttest performances for individual children in the three groups in terms of total score.

Since the children in both groups from school F had received teaching on indices in the intervening period between immediate and delayed posttests, their delayed posttest responses were inspected for errors due to the mis-use of indices in answers that would otherwise be correct (e.g. n^4+p instead of $4n+p$). Table 7.2 shows the data for School F adjusted for the indices error, i.e. answers counted as correct. In each case the appearance of indices in this inappropriate manner was confined to the delayed posttest and was therefore presumably a consequence of the teaching that had occurred between immediate and delayed posttests. This would seem to suggest that more practice in discriminating between the notations '4n' and 'n^4' might have been helpful for these children.

Consideration of the adjusted data shows that for the group of 14-year-old students in School F there appears to have been an improvement in overall performance between pre- and immediate

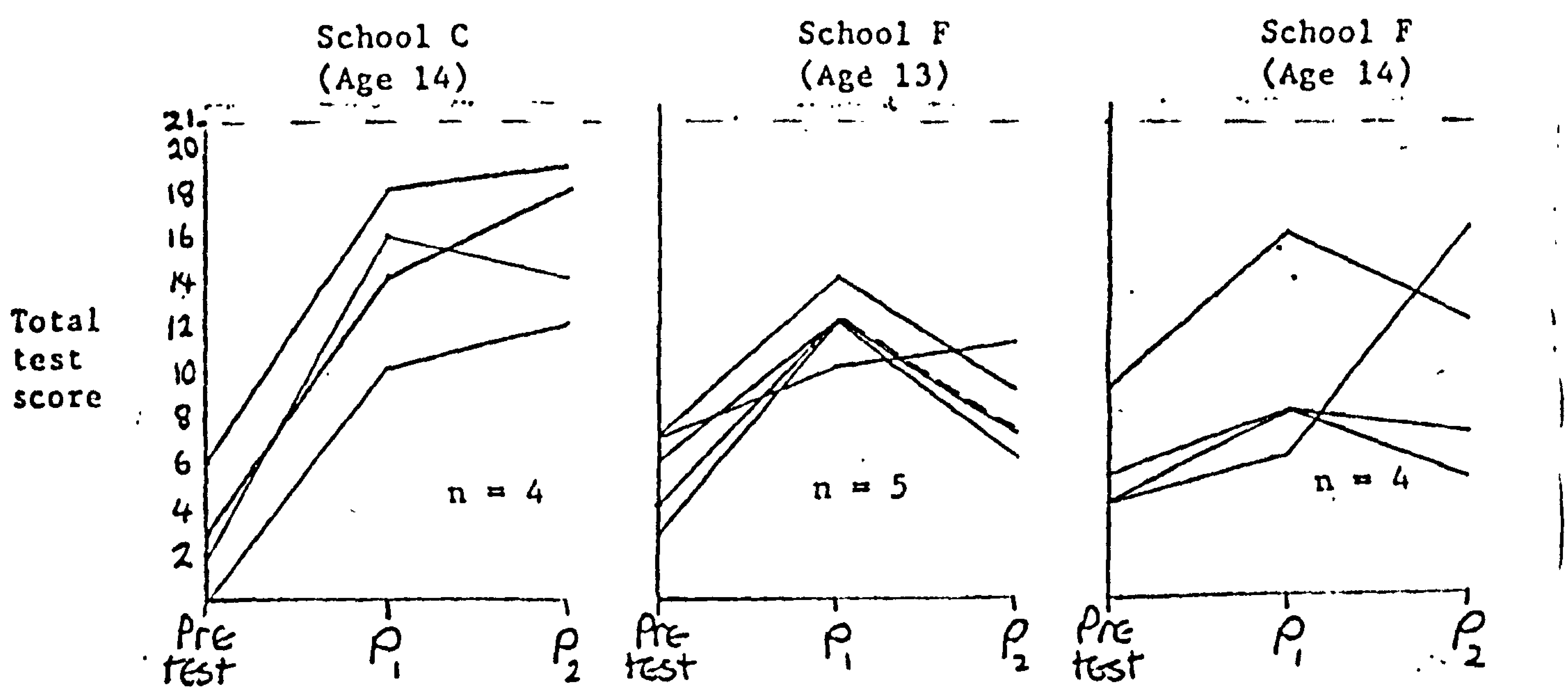


Figure 7.3 Total test scores obtained on the pretest, immediate posttest (P_1) and delayed posttest (P_2) by individual children in the three teaching experiment groups

Table 7.2

Changes in Total Test Score for School F with Delayed Posttest Score Amended for Indices Errors

Year Group	Pupil[c]	Pretest	Immediate Posttest	Delayed Posttest	Amended Delayed Posttest[a,b]
2nd Yr (Age 13)	1	4	12	6	7
	2	6	12	7	8
	3	7	10	11	11
	4	3	12	7	10
	5	7	14	9	13
3rd Yr (Age 14)	1	4	6	16	16
	2	9	16	12	16
	3	4	8	5	8
	4	5	8	7	9

a. Answers involving an 'irrelevant' indices error (e.g. h^4+t instead of $4h+t$) are counted as correct.

b. All tests have a maximum score of 21.

c. Only those pupils who were present for all three tests have been included. This has resulted in a sample of 5 and 4 pupils out of the 2nd and 3rd year groups of 6 and 5 respectively.

posttest, and that this improvement was maintained over the following four months. The results for the group of 13-year-olds (School F) are more equivocal, in that although an improvement between pre- and immediate posttest was observed this was not maintained over the ensuing period, although delayed posttest performance was on average slightly higher than pretest performance. The children in School C received no teaching in algebra in the period between immediate and delayed posttests, and the notably higher level of performance attained by the immediate posttest was generally maintained over this period.

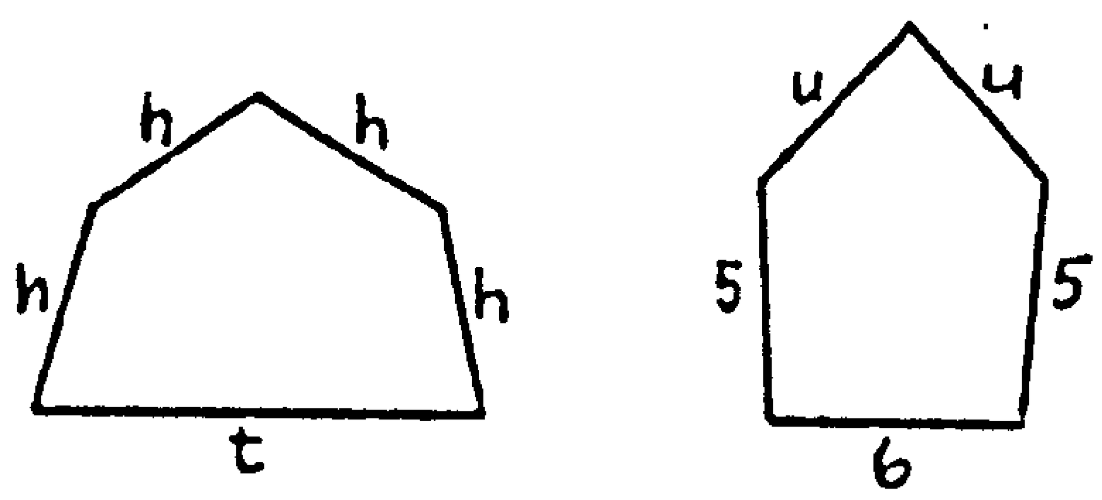
In addition to an examination of overall performance as measured by total test scores, analysis was also made of changes in performance (in terms of number of correct responses and incidence of target errors) on groups of items or on single items suggested to be particularly relevant to each of the identified areas of difficulty. The results of this analysis for the group of 10 items in which conjoining is a possibility are given in Table 7.3 (the actual items are given in Figure 7.4). The results for individual items relevant to the use of brackets, the formalization and symbolisation of method, and the meaning attached to letters (viz. specific unknown or generalised number) are given in Tables 7.4, 7.5 and 7.6 respectively. In each case the analysis includes only those children for whom results on all three tests were available (one child from each group was absent for one or other posttests so that data for these individuals are incomplete). The data in these tables support the observations derived from the teaching experiments (discussed previously) and each area of difficulty is now discussed in turn.

<u>Conjoining</u>. Within the mathematics-machine context, it appeared

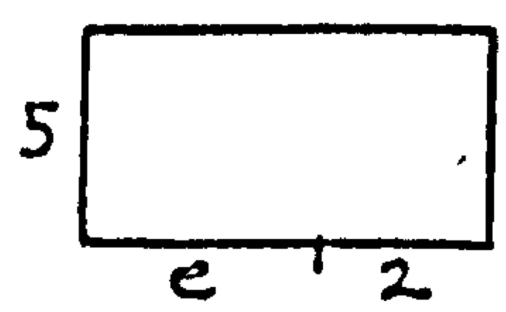
<table>
<tr><th>Item (Abridged)</th><th>Item (Abridged)</th></tr>
<tr><td>

1. Add 4 on to: 3n

2,3. What can you write for the perimeter of:



4. What can you write for the area of:



</td><td>

5. Multiply by 4: n+5

6. If John has J marbles and Peter has P marbles, what can you write for the number of marbles altogether?

Write more simply if possible:

7. 2a+5b

8. (a+b)+a

9. 2a+5b+a

10. If e+f = 8

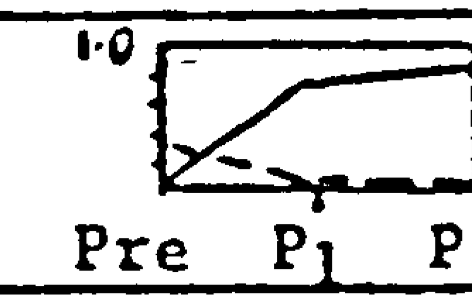
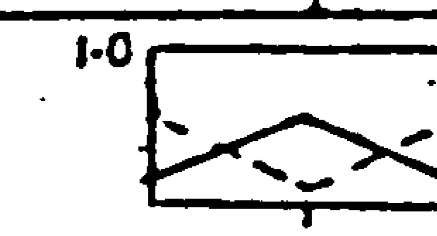
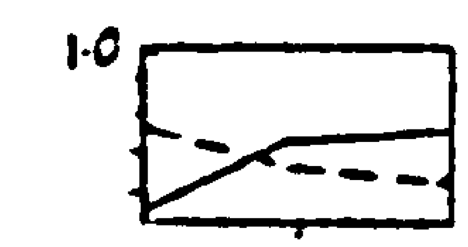
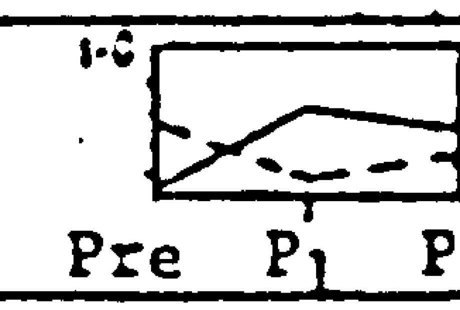
e+f+g = ...

</td></tr>
</table>

Figure 7.4 CSMS test items (used in SESM algebra research) in which conjoining is a possibility

Table 7.3

Changes in Pre- and Posttest Performance on Ten Conjoining Items

School	Year Group	Pretest Number Correct[a]	Pretest Number Errors[b]	Immediate Posttest Number Correct	Immediate Posttest Number Errors	Delayed Posttest Number Correct	Delayed Posttest Number Errors	Change in Performance: Diagrammatic Representation[c]
C (n=4)[d]	3rd Yr (Age 14)	2	14	30	2	32	0	
F (n=5)	2nd Yr (Age 13)	10	29	31	6	10	28	
(n=4)	3rd Yr (Age 14)	4	22	19	14	22	11	
Total:		16	65	80	22	64	39	

a. Number correct refers to the total number of correct answers given by each group of children to the 10 'conjoining' items (e.g. for a group of 4 children the maximum number of correct possible answers is 40).

b. The errors referred to are those in which a conjoined answer was given.

c. Diagram shows proportion of correct answers (solid line) and 'error' answers (broken line) on pretest, immediate posttest (P_1) and delayed posttest (P_2).

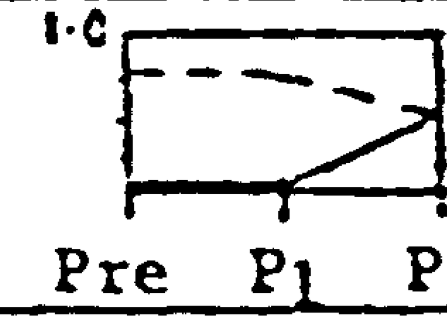
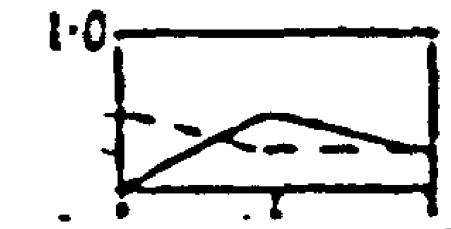
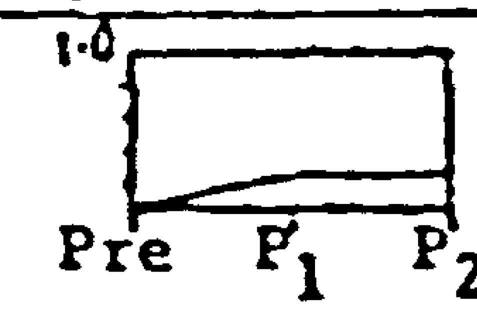
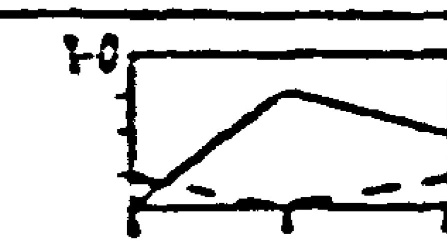
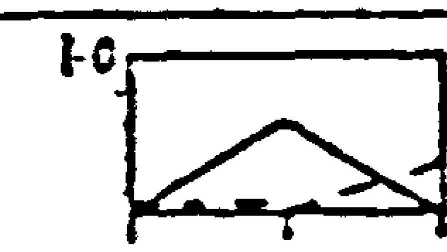
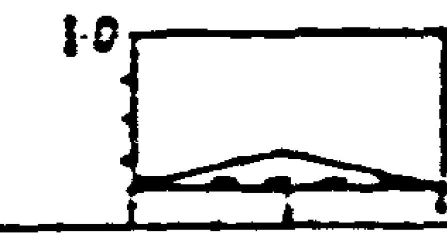
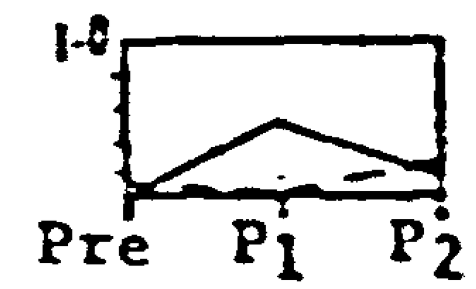
d. Only those children who were present for all three tests have been included. This has resulted in samples of 4, 5 and 4 out of the groups of 5, 6 and 5 respectively.

that children did not find it difficult to accept the notion of indeterminate or non-numerical answers (although several children noted with some surprise that 'the answer is the same as the question'). To the extent that refraining from conjoining reflects an acceptance of this notion, the teaching programme appears to have been reasonably successful in encouraging this acceptance, at least for school C and the school F group of 14-year-olds. In the case of the former, for example, the total number of instances of correct responses over the ten items increased from two to 32 (out of a maximum of 40), while the incidence of conjoined (error) answers decreased from a total of 14 to zero (see Table 7.3). The situation is, however, somewhat different for the school F group of 13-year-olds. While the immediate posttest showed an increase in the number of correct responses and a marked decrease in incidence of conjoined answers for this group, the delayed posttest results signify a regression to the initial state.

Use of brackets. The issue concerning the need for brackets was found to be more complex than initially supposed, and was subsequently made the subject of a separate part of the investigation (discussed later in this chapter). During the teaching experiment the children appeared to readily accept the notion of brackets being needed by the 'machine', but it was by no means clear that recognition of this requirement would be transferred to 'mathematics' or 'algebra' in general.

Table 7.4 shows some modest improvement in recognition of the the need for brackets, but at the same time there is little decrease in the number of 'error' answers, namely those which were apparently correct but for the omission of brackets. As a result, it was decided to amend the teaching programme in order to deal more directly with

Table 7.4
Changes in Pre- and Posttest Performance: Use of Brackets

Item	School	Year Group	Pretest Number Correct[a]	Pretest Brackets Omitted	Immediate Posttest Number Correct	Immediate Posttest Brackets Omitted	Delayed Posttest Number Correct	Delayed Posttest Brackets Omitted	Change in Performance: Diagrammatic Representation[b]
(a) Multiply n+5 by 4	C (n=4)	3rd Yr (Age 14)	0	3	0	3	2	2	
	F (n=5)	2nd Yr (Age 13)	0	1	1	3	0	1	
	(n=4)	3rd Yr (Age 14)	0	2	2	1	1	1	
	Total:		0	6	3	7	3	4	
(b) Area 5 by e+2	C (n=4)	3rd Yr (Age 14)	0	1	3	0	2	1	
	F (n=5)	2nd Yr (Age 13)	0	0	3	0	0	2	
	(n=4)	3rd Yr (Age 14)	0	0	1	0	0	0	
	Total:		0	1	7	0	2	3	

a. Number correct refers to the total number of correct answers per item given by each group of children. For a group of 4 children, the maximum possible correct is therefore 4.
b. Diagram shows proportion of correct answers (solid line) and 'error' answers (broken line) on pretest, immediate posttest (P_1) and delayed posttest (P_2).

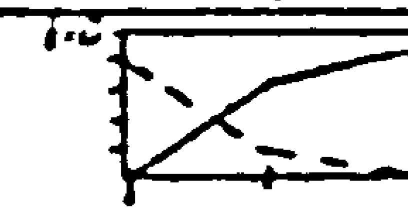
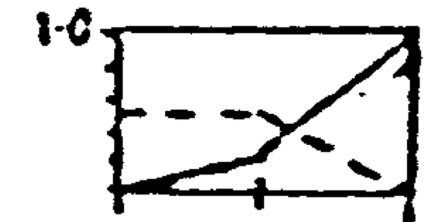
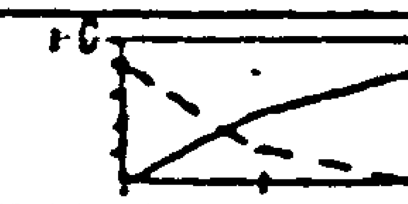
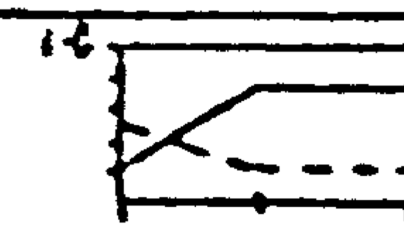
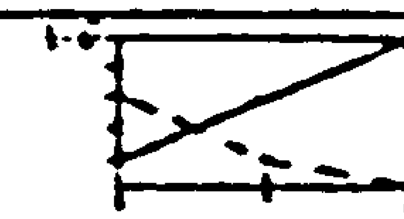
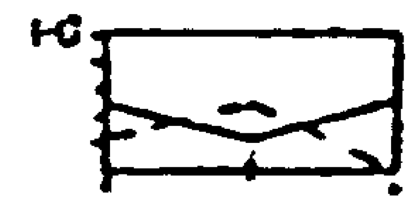
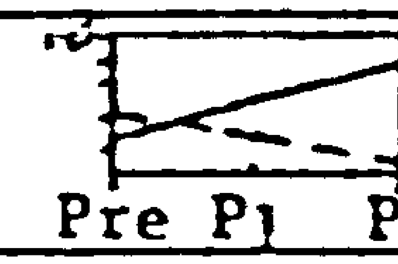
the need for brackets in ordinary arithmetic statements involving more than one operation.

Formalization and symbolisation of method. As in the case of the idea of the need for brackets, the teaching experiment showed that for many children formalizing and appropriately symbolising a procedure for solving even relatively simple problems is not a trivial task. This issue was also studied further in a separate context (described later). However, where the items analysed in Table 7.5 are concerned, the results indicate a significant improvement in success on items of this kind. This was accompanied by a decrease in the incidence of numerical or alphabetical answers, these being suggested to be indicative of a 'lower order' procedure based, for example, on such activities as adding on or counting back.

Meaning attached to letters. A comparison of the results for the two items listed in Table 7.6 reveal a slightly different picture in the two cases. Whilst the results for item (a) show an increase in number of correct responses (i.e. answers giving a correct range of values) and a decrease in the number of single-valued answers suggested to indicate a 'letter-as-specific-unknown', the same pattern is not discerned for item (b), where the incidence of both correct and error answers is relatively constant over the three tests. Part of the reason for this latter result is suggested to be due to an extra factor involved in the second item. Thus while success on item (a) requires a recognition that letters may represent a range of values, success on item (b) requires both this recognition, and acceptance of the notion that a value held by letter is not necessarily thereby excluded from the replacement set of a different letter. Discussions with children during the teaching experiment revealed a possible source of confusion in this regard. In discussing the difference

Table 7.5

Changes in Pre- and Posttest Performance: Formalization and Symbolisation of Method

Item	School	Year Group	Pretest Number Correct[a]	Pretest Number Errors[b]	Immediate Posttest Number Correct	Immediate Posttest Number Errors	Delayed Posttest Number Correct	Delayed Posttest Number Errors	Change in Performance: Diagrammatic Representation[c]
(a) Perimeter n sides, length 2	C (n=4)	3rd Yr (Age 14)	0	4	3	0	2	0	
	F (n=5)	2nd Yr (Age 13)	0	4	3	1	4	0	
	(n=4)	3rd Yr (Age 14)	0	2	1	2	4	0	
	Total:		0	10	7	3	10	0	
(b) Number of diagonals, k-3	C (n=4)	3rd Yr (Age 14)	1	2	3	1	3	1	
	F (n=5)	2nd Yr (Age 13)	1	3	3	1	5	0	
	(n=4)	3rd Yr (Age 14)	2	1	1	2	2	0	
	Total:		4	6	7	4	10	1	

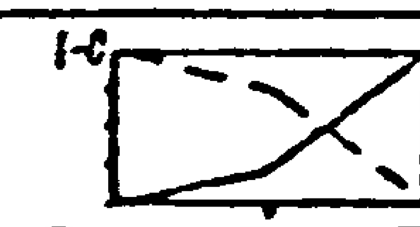
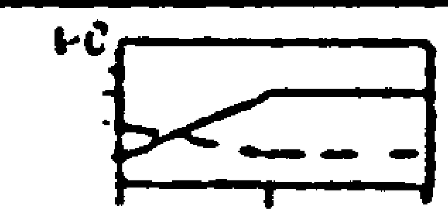
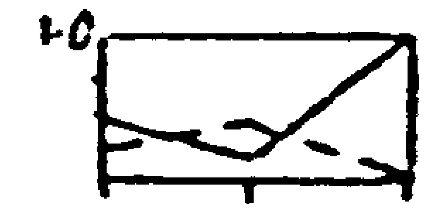
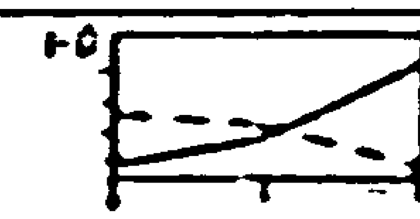
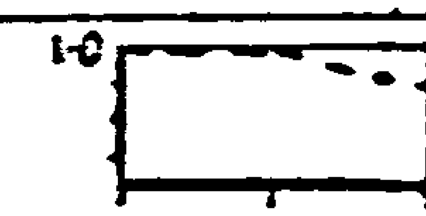
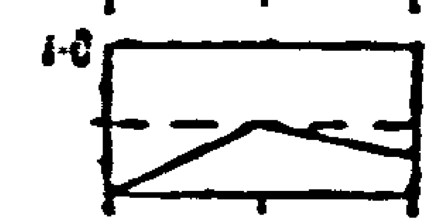
a. Number correct refers to the total number of correct answers per item given by each group of children. For a group of 4 children, the maximum possible number of correct answers is therefore 4.

b. The errors referred to are those in which a numerical or alphabetical answer was given.

c. Diagram shows proportion of correct answers (solid line) and 'error' answers (broken line) on pretest, immediate posttest (P_1) and delayed posttest (P_2).

Table 7.6

Changes in Pre- and Posttest Performance: Interpretation of Letters

Item	School	Year Group	Pretest Number Correct[a]	Pretest Number Errors[b]	Immediate Posttest Number Correct	Immediate Posttest Number Errors	Delayed Posttest Number Correct	Delayed Posttest Number Errors	Change in Performance: Diagrammatic Representation[c]
(a) c+d = 10 and c<d c = ?	C (n=4)	3rd Yr (Age 14)	0	4	1	3	4	0	 Pre P_1 P_2
	F (n=5)	2nd Yr (Age 13)	1	2	3	1	3	1	
	(n=4)	3rd Yr (Age 14)	2	1	1	2	4	0	
	Total:		3	7	5	6	11	1	
(b) L+M+N =L+P+N?	C (n=4)	3rd Yr (Age 14)	0	4	0	4	0	3	
	F (n=5)	2nd Yr (Age 13)	0	4	0	2	2	2	
	(n=4)	3rd Yr (Age 14)	0	2	2	2	1	2	
	Total:		0	10	2	8	3	7	 Pre P_1 P_2

a. Number correct refers to the number of correct answers per item given by each group of children. For a group of 4 children, the maximum possible number correct is therefore 4.

b. The errors referred to are those in which a single-valued answer (item 'a') or the answer 'never' (item 'b') was given.

c. Diagram shows proportion of correct answers (solid line) and 'error' answers (broken line) on pretest, immediate posttest (P_1) and delayed posttest (P_2).

between, say, a + a and a + b, the point had been made that in the former case the values of the two letters had necessarily to be identicial, whereas in the latter case the values could be the same or different. It seems that some children interpret 'or' in such situations in an exclusive, rather than inclusive sense. Remarks to the effect that 'the values can be the same or different, i.e. you have a choice' in fact reinforced this notion by suggesting that they could choose the values to be <u>either</u> the same <u>or</u> different, but not both. Even for children who did understand the inclusive nature of the argument, the idea of the same values occurring for different letters appeared to be somewhat unlikely:

> RD (14 years, during teaching experiment): Yes, well I know they <u>could</u> be the same (i.e. the values of a and b), but most of the time they'll be different, won't they? You won't get them turning up the same very often.

A similar difference in interpretation was observed for use of the word 'any'. Consideration of 'generalisation' items of the kind 'add 3 to any number I give' initially produced responses from some children such as '3 + 100'. Whereas the teacher-experimenter viewed 'any' as meaning 'all possible values in the world', many of the children interpreted it as meaning 'any particular one you choose' (which is, of course, its more common everyday meaning). It was consequently thought necessary to address both these issues more carefully in the subsequent teaching.

An observation concerning the values ascribed to letters was also of interest. The majority of 'correct' answers to item (a) (Table 7.6) comprised sets of discrete integer values (all positive). Even though the children were familiar with the use of inequality symbols, which would enable them to symbolise a range which included non-integral values, the evidence from the teaching experiment clearly showed that they typically confine themselves to whole-numbers. This point was clear from the CSMS findings, and had in fact been reiterated by the description derived by SESM of the general nature of informal 'child-methods' (see Chapter 6). Since the teaching programme had also largely ignored the existence of negative or non-integral values (despite familiarity with the above findings!), it is perhaps not surprising that children do likewise. This issue was given special attention in the later teaching.

Gains in delayed posttest performance. An additional point arising from the results of the teaching experiment relates to the observed gains in performance on some items between the immediate and delayed posttests (see diagrammatic representations in Tables 7.5 and 7.6 in particular). If the teaching programme had succeeded in its aim of allowing children to restructure their thinking with regard to the various aspects under consideration, rather than to merely acquire knowledge which might be readily forgotten, then such an improvement would be expected. The observation of such a trend may lend some support to the view that the teaching programme was successful in achieving this aim for a large proportion of the children involved.

Further Investigation: The Use of Brackets

The issue concerning children's views on the need for brackets appeared, from both interviews and teaching experiments, to be unexpectedly complex. Consequently, it was thought useful to examine this aspect further. This was done by means of the analysis of selected items from a paper-and-pencil test. These items were designed to assess children's recognition of the need to use brackets in algebraic statements and their perception of equivalent expressions in this domain (see Figure 7.5 for a sample item from the test; the test itself is included in Appendix 8). In addition, information was also obtained by discussions with mathematically-able pupils as well as pupils of average mathematical ability from the teaching experiment groups.

A total of 991 children aged from 13 to 16 years from the full ability range and from three schools (not previously used in the study) were given the test. Table 7.7 shows the results for the item illustrated in Figure 7.5. From this it can be seen that a large proportion of children in each year group appeared to regard brackets as irrelevant in that they considered expressions with and without brackets to be equivalent. Also of interest is the fairly high proportion of children who excluded bracket statements from their answers. Inspection of the data analysed by ability groups (Table 7.8) shows that both observations were not confined to the lower ability groups, but that children in the top ability groups also appeared to ignore the need for brackets. Since work during the teaching experiments had suggested that children in middle-ability mathematics groups knew about brackets but did not consider their use necessary, and so did not use them, it was thought useful to see if this was also the case with the mathematically-able pupils. This question was investigated in a series of 15 interviews with children aged 13 to 16 years from the top streams of two schools and who had been selected by their teachers as being particularly able at mathematics.

As in the case of children from middle-ability groups, children from these high ability groups also appeared to be familiar with the bracket notation, but considered their use to be largely optional. This belief appeared to be founded on three main assumptions (see also

1. Which of the following can you write for the area of this rectangle?
   Tick <u>every</u> one you think is correct:

   5 x e+2 ......

   5 x (e+2) ......

   10e ......

   5 x e2 ......

   5(e+2) ......

   e + 2 x 5 ......

   None correct ......

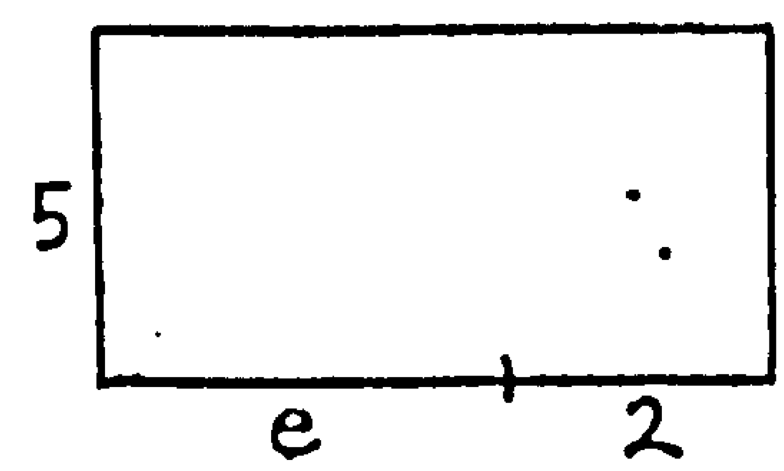


Figure 7.5 Example of item designed to test children's recognition of the need for brackets (item also includes test of children's acceptance of the conjoined term)

Table 7.7

Number of Children Making Correct and Incorrect Responses to Illustrative Item from Paper and Pencil Test on the Use of Brackets and Conjoining

Item Response	Year Group 2nd Yr (Age 13)	3rd Yr (Age 14)	4th Yr (Age 15)	5th Yr (Age 16)
Expressions for area 5 by e+2:				
Correct	6(8%)[a]	29(10%)	80(20%)	82(34%)
Brackets and non-brackets equivalent	34(32%)	105(37%)	184(46%)	92(38%)
Brackets excluded	20(25%)	77(27%)	63(16%)	37(15%)
5 x e2 included	29(37%)	90(31%)	75(19%)	38(16%)
Number Tested:	79	286	397	239

a. Percentages do not total 100 since response categories are not mutually exclusive, e.g. a child may both exclude brackets and include the conjoined answer 5xe2.

Table 7.8

Number of Children at Different Levels of Mathematical Attainment[a] Making Correct and Incorrect Responses to Paper and Pencil Test Item on the Use of Brackets

Year Group	Response[b]	Mathematics Ability Band		
		Top	Middle	Lower
3rd (Age 14)	Correct	11 (12%)	10 (10%)	8 (9%)
	Brackets and non-brackets equivalent	40 (43%)	37 (37%)	28 (30%)
	Brackets excluded	18 (19%)	28 (28%)	31 (33%)
	Other	24 (26%)	25 (25%)	26 (28%)
4th (Age 15)	Correct	52 (36%)	18 (13%)	10 (9%)
	Brackets and non-brackets equivalent	63 (43%)	75 (53%)	46 (42%)
	Brackets excluded	16 (11%)	22 (15%)	25 (23%)
	Other	15 (10%)	27 (19%)	28 (26%)

a. Data given for 3rd year and 4th year age groups only.

b. Response given to item shown in Figure 7.5

Booth, 1982d, see Appendix 13):

1. you perform a string of operations in the order written (left to right computing);
2. in cases where a context is given, this determines the order in which the operations are to be performed, regardless of the order in which they are written (influence of context);
3. the same answer will in any case be obtained regardless of the order in which a sequence of operations is computed (associability of operations).

The following illustrations from the interviews with the children designated as mathematically-able demonstrate these viewpoints, and are remarkably similar to those provided by children from the lower ability groups:

1. Left to right computing. P (12 years), explaining the difference between 18+27x19 and 18x27+19 (I: Interviewer):

   P: Well, they've got the add sign and multiply sign in a different order. That one (18+27x19) says add first, and this one (18x27+19) says multiply first.
   I: How does it say that?
   P: By the order it's written!

   K (13 years), explaining his choice of e+2x5 for the area of the rectangle measuring 5 by e+2:
   K: 'Cos you add those together first (e+2) and so you get the length.
   I: And how do you know to add first?
   K: 'Cos it comes first!

2. Influence of context. R (15 years), explaining her choice of 5xe+2 for the area of the rectangle measuring 5 by e+2 in terms of the diagram given:

   R: Well, you don't know the e, so that length is e+2, and then you've got to multiply this side by the width and that's 5 ..... so it's 5 times e plus 2.

I: And that tells me to add the e and 2 and then times it by 5?
R: Yes, you need the e plus 2 first to give you the whole side (indicates on diagram).

3. Associability of operations. K (13 years), computing 18x27+19, having just calculated 27+19x18 from left to right:

K: Do... 27 plus 19, then multiply by 18. It's the same as the last one ..... it's just the other way round.
I: Right, well, suppose I came along and thought it meant multiply 18 by 27, and <u>then</u> add 19. Would I get the same answer?
K: Yes
I: Which way would <u>you</u> do it?
K: Either! Either way. Depends what comes into my mind at the time.
I: But would it matter which way you did it?
K: No, you'd still get the same answer.

Since these viewpoints appeared to be strongly held, it was considered that any attempt to persuade children to use brackets would be unrewarding unless children were first able to perceive that computing expressions in different orders did in fact lead to different results, and that some convention was therefore necessary to indicate which order (and solution) was intended.

Amendments to the Teaching Programme

On the basis of the above findings and the observations made during the small-scale teaching experiments, various amendments to the teaching programme were suggested. These are summarised under the appropriate teaching programme component headings (described previously in the section 'Outline of the Programme'):

	Step	Outline of Content
1.	Introduction	The inclusion of non-integral values in all class examples and worksheets, and the encouragement of replacement values other than counting numbers throughout the programme where applicable.
2a.	Notation-Brackets	The inclusion of a section to point

out the importance of brackets in mathematics in general as well as for the 'mathematics machine'. It was decided, for present purposes, to over-ride the conventions for order of operations, and to introduce the notion that brackets would be required in order to indicate priority of operation.
Example a: 3x2+4. 'Calculate two ways, doing the 'add' first, and doing the 'times' first. Will you get the same answer?'
Example b: 'Put in brackets so as to arrive at the answer given':
4x5+2= 28

3.	Generalisation I	Discussions of instructions for the general case 'add 3 to any number'. Discussion of the meaning of 'any', and of the difference between, for example, 'n+3' and '18+3'. Discussion of possible replacement values for different letters, emphasizing that equal values can be ascribed to different letters. Encouragement of consideration of this case in pupil-generated replacement sets.
4.	Generalisation II	No amendments.
5.	Notation	The inclusion of further practice examples to provide additional reinforcement. Provision of a 'rationale' for the abbreviation 3 x a to 3a in terms of providing a clearer distinction between the otherwise perceptually similar expressions 3 x a and 3 + a.
6.	Consolidation	No amendments.

The teaching programme was consequently amended in the manner outlined above, and this second version used in the pilot class teaching trials to be described in the next chapter. Examples of the amended material are given in Appendix 4. Otherwise the worksheets used in the class pilot study were the same as those used in the small scale teaching experiments (shown in Appendix 3).

Since the 'dialogue' teaching approach adopted as a means of permitting an on-going monitoring of the teaching-learning process, as required by the need to achieve some form of 'direct measurement' (see Chapter 3), had proved to be extremely useful also as a teaching device, it was decided to build this approach into the teaching programme.

CHAPTER 8: CLASS TRIALS

Before assembling the teaching module into a form which could be used by teachers with their own classes, a study was conducted in which the researcher taught the programme in the class setting with groups of approximately 15 children in each of four age groups from 12 years (first year) to 15 years (fourth year). The major aim of these trials was to check the workability of the teaching programme in the normal class situation, and to make any amendments to the teaching materials which might be suggested to be necessary for use with other teachers. Following this, a teaching module 'kit' was prepared which comprised a set of detailed teaching notes and class sets of worksheets and answer sheets. Volunteer teachers were then recruited to try the programme with different age groups of children, and briefed on the presentation of the programme and use of the material. As in the case of the first class trials by the researcher, these 'other teacher' class trials were evaluated using the same pretest, immediate posttest and delayed posttest procedure outlined in connection with the small-scale teaching experiments (Chapter 7).

Class Teaching (Researcher)

Sample

One class from each of the first to fourth year secondary school levels (ages approximately 12, 13, 14 and 15 years respectively) from one school was used in this part of the study (the school used was the co-educational comprehensive school 'D' used in the interview phase of the research programme). These classes were taken from one teacher's timetable and covered a range of ability levels. A brief description (giving age, level and number of children) of each class is given in Table 8.1, 'CSMS Number Tests - Results'. Except for the group of 12 year-olds, each class was divided into two approximately equal groups on the basis of their performance on the CSMS Algebra test given as pretest as described in Chapter 7. The children chosen for inclusion in the algebra group were those best meeting the criteria (described in Chapter 5) concerning the kinds of error made. The 12 year-old children had received no specific teaching in algebra and this class was not given the CSMS Algebra pretest, but was randomly divided into two groups. The children not allocated to the algebra groups formed the sample for the SESM class trials (researcher) in the ratio programme.

Because the question of the child's ability to represent formal mathematical method had been suggested in the interviews to contribute to the child's understanding of algebra, in the sense of generalised arithmetic, it was decided to administer the CSMS Number Operations Test or part of the CSMS Decimals Test as well as the Algebra pretest to the classes participating in the class teaching (researcher) phase of the algebra research. The sections of the CSMS tests used were those requiring the child to select from a given set of arithmetical expressions those which were appropriate to the solution of a given word problem (see Figure 8.1 for examples of the items in question).

Procedure

The class teaching (researcher) phase of the research proceeded in the manner already described for the small-scale teaching experiments, using the amended programme described in chapter 7, but with one modification. In the case of the first year group (age 12 years) who had received no previous teaching in algebra and the third year group (age 14 years) which was particularly weak in mathematics, the section of the programme on the more purely

Test	Item		
CSMS Number Operations	A gardener has 381 daffodils. These are to be planted in 23 flowerbeds. Each flowerbed is to have the same number of daffodils. How do you work out how many daffodils will be planted in each flowerbed?	391 − 23 23 − 391 391 + 23 23 x 17	23 ÷ 391 391 x 23 23 + 23 391 ÷ 23
CSMS Place Value and Decimals	My car can go 41.8 miles on each gallon of petrol on a motorway. How many miles can I expect to travel on 8.37 gallons?	41.8+8.37 41.8÷8.37 41.8−8.37	8.37÷41.8 8.37 −41.8 8.37x 41.8

Figure 8.1 Examples from CSMS Number Operations Tests and Place Value and Decimals Test.

notational aspects of algebra in the teaching module was omitted, and items of the 'simplify 2a+5b+a' type excluded from the posttests given to these groups. The abbreviated posttests thus administered to the first and third year groups are given in Appendix 9.

The instructional phase commenced in the week following the administration of the pretest, and comprised five to six 40 minute lessons, conducted in a period of six consecutive days. The immediate posttest was administered in the lesson following the final teaching period and the interval between immediate and delayed posttests was eight weeks and included the Christmas vacation. Little algebra teaching was done during this time, although the fourth year group (age 15 years) did revision which included algebra, and the third year group (age 14 years) did some work on elementary equations of the form x+2=5.

Observations

Use of brackets. Two observations in particular may be made concerning the children's response to different sections of the teaching programme. Firstly, the apparent resistance of children to the idea of using brackets has already been commented upon, as has the complexity of the factors which seem to underlie this resistance. The teaching module had been amended to include specific work intended to alert children to the need to define the order in which operations should be performed. However, it quickly became apparent that this was too large an issue to be satisfactorily dealt with in the short space of time (and material) allocated to it. Children were convinced that strings of operations should be performed in the order in which they were written (see, for example, Worksheet 3 in Appendix 4). They were also generally convinced that the same answer would be obtained regardless of order of computation; the demonstrated invalidity of this notion (achieved by actually calculating the same expression in different orders), was something they found hard to fully accept. While activity of this kind was useful in focussing children's attention on the problem, it was not in itself sufficient to resolve the difficulty. Since the necessary time could not be devoted to this issue within the confines of the present teaching programme, the decision was made to omit this sequence from future work on the module, and to revert to the initial procedure by which the need for brackets was introduced as a machine requirement.

Representation of method. The second difficulty which arose related to children's ability to model word problems by the appropriate arithmetical expression (regarded as being a precursor to the ability to represent the structure of problems in the algebraic case). One of the first pieces of work that children meet in the teaching module requires them to write the 'instructions' (or arithmetical expression) appropriate to the solution of simple word problems. That this was not a trivial matter had been observed during the small-group teaching experiments, but it was not until the programme was tried with the larger groups of children used in the class teaching (researcher) study that the scope of this difficulty became fully apparent. The results from the preliminary testing on the CSMS Number Operations and Decimals Tests (Table 8.1) showed that many of the children had difficulty with this kind of question, with only five out of the 114 children tested getting every item correct, and 94 children making two or more errors (out of six or nine items according to whether the test used contained decimal or whole number values). (See also Brown, 1981c; Brown and Kuchemann, 1976.)

These results were supported by the observations made during the

Table 8.1

CSMS Number Tests - Results

Year[a] Group	Level	CSMS Test	Number of Items	Number of Children Tested	All Correct	2 or More Wrong
1st Year (Age 12)	Mixed Ability	Number	9	30	1	27
2nd Year (Age 13)	Top Stream	Decimals	6	32	1	23
3rd Year (Age 14)	Lower Middle Stream	Number	9	22	0	22
4th Year (Age 15)	Middle Stream (CSE)	Decimals	6	30	3	22
		Total		114	5	94

a. The school involved was school 'D' (see Table 2.2) - Coed. Comprehensive (Middlesex).

teaching programme when it became apparent that a large proportion of children did not find this task easy, and also that they made a substantial number of errors in recording, especially when division or subtraction was involved. A possible basis for the observed error in recording division and subtraction is outlined in Booth, 1982d (see Appendix 13). These observations, of course, support the evidence derived from the interviews that the representation of formal method in mathematics is something which does not come readily to many children.

Integral versus non-integral values. Initially all the examples which the children produced were concerned with whole number values. It was necessary to make a conscious effort throughout the programme to encourage children to suggest non-integer examples. Towards the end of the programme, however, children (especially the younger ones) appeared to be entering into the spirit of things and values such as three million two hundred and eighty-three point nine seven became a regular occurrence.

The interpretation of letters. The notion that letters could represent a range of values and that different letters could be assigned the same value seemed to be a very intriguing idea for many of the children, particularly those in the first and second years (ages 12 and 13). The fourth year pupils were more sceptical about this, and reiterated that the whole point of having different letters was that the numbers represented should also be different. When their attention was drawn to the case of the graph $y = x$, they explained that 'this was different', as 'this was graphs' and had nothing to do with algebra. This group was also very resistant to the idea that a letter could be regarded as representing a range of values rather than just one single, if as yet unknown, value. Both of these (related) viewpoints appeared to be based on previous experience, and were reflected in the results of this group on the items testing level of letter interpretation (see Figure 8.6 later in the chapter), where virtually the same low level of performance was obtained across the three pre- and posttests.

Where the children in the first and second year groups were concerned, the discussion on the use and meaning of letters was considerably facilitated by the mathematics machine context and in particular by the use of the 'model'.

Test-Results

Overall performance. Changes in total test score obtained on the pre- and posttests by individual members of the second, third and fourth year groups, and changes in performance between the two posttests in the case of the first year group, are shown in Figure 8.2.

The results for pre- and immediate posttests for the three groups concerned show a significant improvement in performance between the two tests, and this improvement is generally maintained between immediate and delayed posttests (see Tables 8.2 and 8.3).

The results for the first and second year groups are interesting in that they indicate an actual improvement in performance between the immediate and delayed posttests on the part of a large number of children. In the case of the first year group (who had received no previous teaching in algebra), this increase was significant and was demonstrated for all the children except one, this child obtaining the same score on both tests (see Figure 8.2 and Table 8.3). The absence of any algebra tuition during the intervening period between the two posttests makes it hard to attribute this improved performance to the

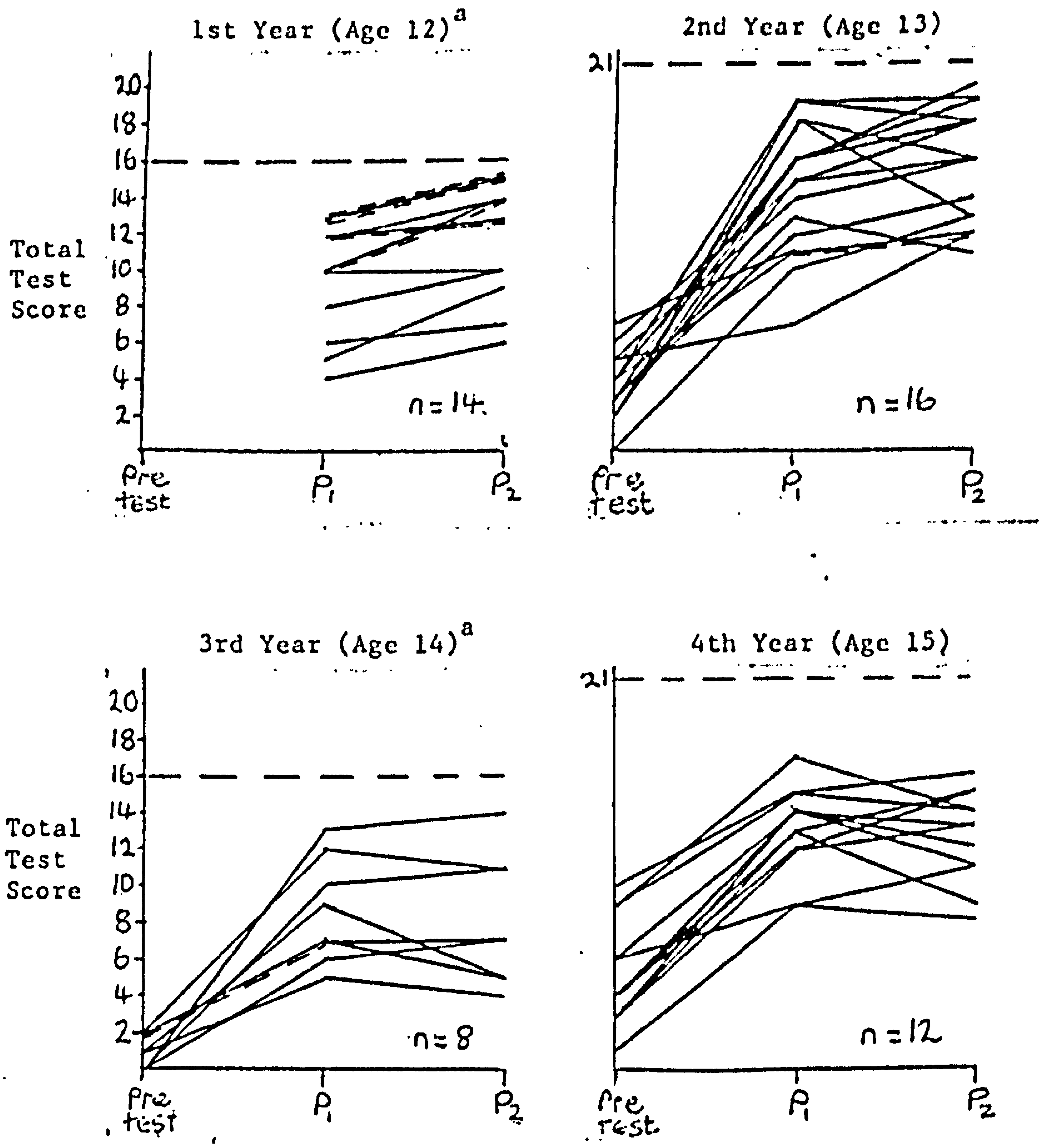


Figure 8.2 Total test score obtained on the pretest, immediate posttest (P_1) and delayed posttest (P_2) by individual children in the four class teaching (researcher) groups.

a. Maximum score on the pre- and posttests for the 1st and 3rd year groups was 16 as compared with 21 for the 2nd and 4th year groups.

Table 8.2

Mean Scores for Pre- and Posttests
for the Four Class Teaching (Researcher) Groups

Year Group	Maximum Score[a]	Pretest	Immediate Posttest	Delayed Posttest
1st Year (Age 12)	16	-	10.00	12.0
2nd Year (Age 13)	21	3.5	14.4	15.4
3rd Year (Age 14)	16	1.0	8.6	8.0
4th Year (Age 15)	21	5.2	13.1	12.6

a. The maximum score obtainable on the tests was lower for the first and third year groups as those items of a more notational nature (such as 'simplify 2a + 5b + a') were excluded for these children (see text).

Table 8.3

Pre- and Posttest Performance for the Four Class Teaching (Researcher) Groups

Year Group	Maximum Score[a]	Mean Gain[b]	s.d.	N	t	df	p[c]
Pretest vs Immediate Posttest							
2nd Year (Age 13)	21	10.87	4.27	16	9.86	15	$p<.001$
3rd Year (Age 14)	16	7.62	3.16	8	6.38	7	$p<.001$
4th Year (Age 15)	21	7.92	2.27	12	11.57	11	$p<.001$
Pretest vs Delayed Posttest							
2nd Year (Age 13)	21	11.94	3.59	16	12.88	15	$p<.001$
3rd Year (Age 14)	16	7.00	4.04	8	4.58	7	$p<.005$
4th Year (Age 15)	21	7.42	2.71	12	9.08	11	$p<.001$
Immediate vs Delayed Posttest							
1st Year (Age 12)	16	2.00	1.24	14	5.82	13	$p<.001$

a. The maximum score obtainable on the tests was lower for the first and third year groups as those items of a more notational nature (such as 'simplify 2a + 5b + a') were excluded for these children (see text).

b. Gain scores are obtained by subtracting the pretest score from the posttest score for each individual.

c. Values of p are for a one-tailed t-test.

effects of experience or practice with the kind of items under study. If the effect of the teaching programme had been to help the children to structure more appropriately their thinking, and if time were required in order to permit such structuring to proceed, then one might predict an improvement in understanding over time which would presumably be reflected in improved performance. It is suggested that the improved performance of those individuals concerned does in fact reflect an improved level of understanding brought about in this manner.

A special word is due concerning the performance of the third year group. This group was considered to be very weak in mathematics, and their performance on the algebra pretest was perhaps not surprisingly very low (see Figure 8.2, also Table 8.2). As a consequence, the last part of the teaching programme, which is concerned with the more abstract notational aspects of elementary algebra, was omitted with this group, as indicated in the discussion on the class trials (researcher) procedure. This also permitted more time to be spent on the earlier part of the programme. Although the final level of performance of this group (as measured by mean delayed posttest score, see Table 8.2) leaves much room for improvement, the results are nevertheless encouraging in terms of the significant improvement in delayed posttest performance as compared with pretest performance. The results also showed that this group's delayed posttest performance was better than the pretest performance of both second year (top stream) and fourth year (CSE stream) groups, each of whom had studied algebra before. From this point of view it is perhaps not wise to assume that elementary algebraic notions are <u>necessarily</u> too difficult for children in lower streams (although the teachers of such children may well decide to omit teaching this topic for other reasons).

A breakdown of changes in performance for individual items (or groups of items) which may be regarded as measuring understanding of the various areas of difficulty identified earlier in the study, provides some evidence of the efficacy of the teaching programme in addressing those different difficulties. Since several conceptual aspects may contribute to success or failure on a given item, it is not always possible to pinpoint a particular item as measuring only one given conceptual area. For example, the CSMS 'marbles' item ('If John has J marbles and Peter has P marbles, what can you write for the number of marbles they have altogether?') requires both the ability to represent the solution by the correct mathematical expression (in this case the sum J + P), and the ability to appreciate that this is an appropriate thing to do, and that the expression represents both a 'method' and an 'answer' so that there is no need to seek a single-term, or conjoined, answer. Consequently, this item cannot be regarded as measuring solely the 'formalization and symbolisation of method' aspect, or that relating to 'conjoining'. Indeed, it is likely that these two aspects are themselves related, as discussed earlier. Nevertheless, on the basis of the kind of error mainly associated with an item, several items from the tests have been selected as providing measures of understanding primarily in one area or another. These items are therefore used to provide an indication of the changes in children's level of understanding in the various areas of difficulty described earlier, namely:

a) conjoining in algebraic addition,

b) the non-use of brackets,

c) the meaning attached to letters (especially the interpretation of letters as representing specific unknowns rather than generalised number), and

d) the formalization and symbolisation of method.

The changes in performance on the items suggested to be most relevant to each of these areas are illustrated in Figures 8.3 and 8.4 for the first and second year groups respectively, and in Figures 8.5 and 8.6 for the third and fourth year groups (these Figures appear later in the chapter). Items specifically relating to the difficulty in formalizing and symbolising method are less readily singled out, since many of the items in the test may be regarded as being associated with this aspect to one degree or another. However, a few items have been selected as being particularly relevant to this issue, and these are also indicated in Figures 8.3 to 8.6. The interview findings suggested that children who have difficulty either with the actual representation of method, or with perceiving that such representation may be appropriate, typically seek a numerical or alphabetic answer. However, they may also produce an incorrect attempt at representing the correct method, such as by writing -3k for 'subtract 3 from k'. Consequently, answers in either category have been grouped together to represent errors of 'formalization and symbolisation of method' in the analysis depicted in Figures 8.3 to 8.6.

Performance by item: first and second year groups. As the first year group (age 12) did not receive the pretest, only posttest results are available for this group (see Figure 8.3). These results show that the items, and hence the associated conceptual areas, can be grouped into two sets according to the level of understanding (as measured by the level of success on an item and the incidence of competing 'error' answer) attained on the posttests. Those items primarily associated with a tendency to produce a conjoined answer

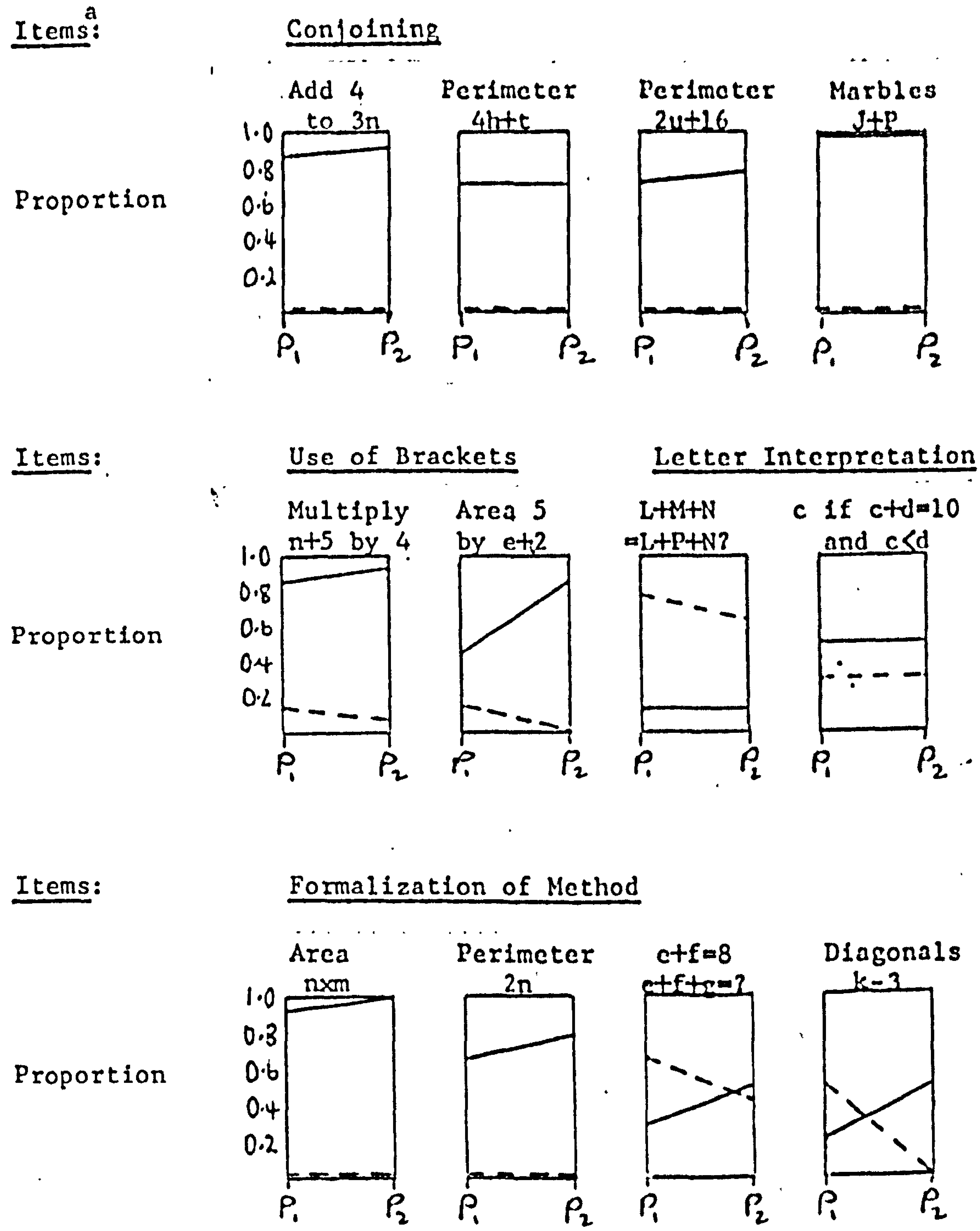


Figure 8.3 Proportion of children (n=14) giving correct (solid line) and 'error' (broken line) answers to individual items on immediate (P_1) and delayed (P_2) posttests for 1st year (age 12) class teaching (researcher) group.

a. Brief descriptions only of each item are given here. The full items appear in the pretest given in Appendix 6.

(alternatively viewed as those items reflecting the child's willingness to accept an unclosed answer) and those requiring the use of brackets show a high level of success accompanied by a very low or zero incidence of the related error. The absence of conjoining in any item was particularly notable, given the prevalence of this error observed elsewhere.

A lower level of performance and a higher error incidence is observed for the level of letter interpretation items and for those associated with the representation of method. This would seem to suggest that the teaching programme was less effective in addressing these two aspects with this group of children. However, some of the representation of method items are of interest in that they reveal an improvement in performance between the immediate and delayed posttests (see Figure 8.3). Among these items may perhaps be included the use of brackets question concerning the area of a rectangle measuring 5 by e+2. This item is suggested to reflect as much a 'formalization of method' aspect as a 'use of brackets' aspect, and indeed it can be seen from Figure 8.3 that the improvement in performance on this item between immediate and delayed posttests is greater than can be explained by the drop in incidence of bracket-omission errors. An improvement in performance of this kind might be anticipated if children were taking time to construct an appropriate cognitive viewpoint with regard to the ideas concerned. Consequently, while the teaching programme appears not to have been effective in stimulating an immediately high level of success on these items, it may have been effective in initiating a process which leads to improved understanding of the related issues, but which takes time to reach fruition.

Apart from these items, the results generally show virtually no change in facility between the two posttests. This would seem to imply that acceptance of the notions associated with the other items (namely those relating to conjoining, the use of brackets and the level of letter interpretation) proceeds in an 'all or none' fashion, in that children either accept (and assimilate) the ideas when first introduced, or they do not. The fact of assimilation is indicated by the maintained level of performance over the two month interval between posttests. Thus the children appeared not to forget the principles involved, as might have been expected if the ideas had been more superficially received.

The results for the second year group of children (age 13) similarly showed some high levels of performance and low levels of error incidence on the two posttests (see Figure 8.4). Where the items associated with conjoining are concerned, the posttest level of performance is in fact very similar for the first and second year groups, despite the apparent difference in mathematical ability between the two groups (the second year group was selected from the top mathematics stream whereas the first year group was randomly selected from a mixed ability class).

Results for the 'use of brackets' items reveal a difference in favour of the first year group. However, it should be remembered that the first year group did not meet that part of the teaching programme dealing with simplification of algebraic expressions, whereas the second year children did. As a result, the first year group were writing all operations in full (e.g. 3xa and 2x(a+4) instead of 3a and 2(a+4)), and so may have been less disposed towards a possibly incorrect abbreviation of answers. This may also, of course, explain (at least in part) the marked absence of conjoined answers noted for the first year group.

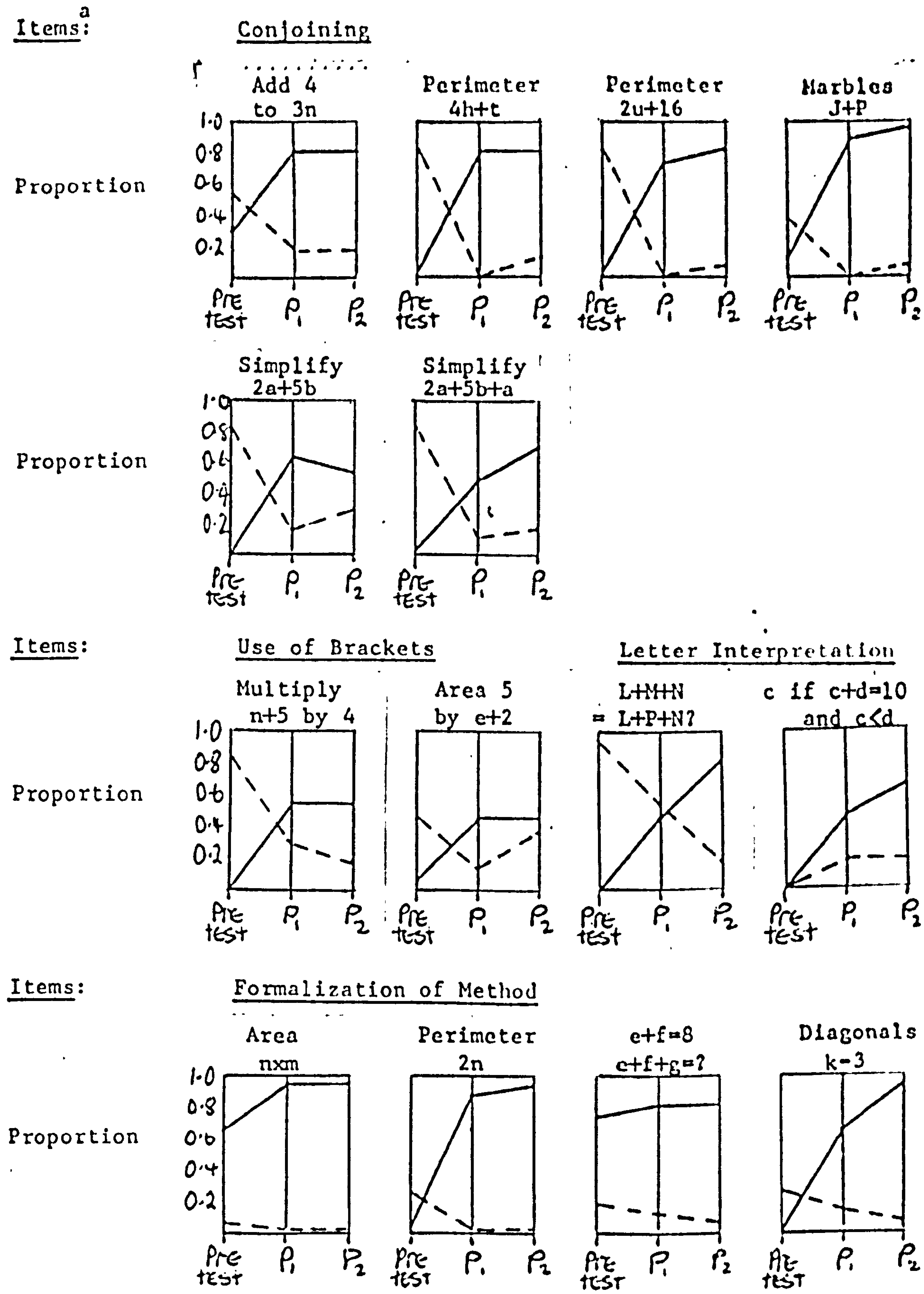


Figure 8.4 Proportion of children (n=16) giving correct (solid line) and 'error' (broken line) answers to individual items on pretest, immediate posttest (P_1) and delayed posttest (P_2) for 2nd year (age 13) class teaching (researcher) group.

a. Brief descriptions only of each item are given here. The full items appear in the pretest given in Appendix 6.

The second year group appear, however, to have attained a posttest level of performance on the representation of method items and also on the level of letter interpretation items which is somewhat higher than that achieved by the first year group. The latter items in particular are of interest in indicating the same kind of delayed improvement between posttests as was noted for the first year group in the case of the formalization of method items. It seems that the same programme which had apparently little effect on the first year children's level of letter interpretation (in terms of their appreciation of letters as generalised number rather than specific unknowns) had a much more pronounced influence in this regard on the second year children's level of understanding, as measured by the increase in the number of children successfully answering the items relevant to this issue. This may indicate the operation of a 'readiness' factor, by which assimilation of a given concept requires a particular level of cognitive maturation in addition to the establishment of an appropriate contextual framework. One might conjecture that the pattern of performance obtained over time may provide information on the 'matching' of item difficulty with children's level of cognitive functioning. Viewed from this perspective, an immediate high gain in performance on a group of items, with little subsequent growth, may indicate that the ideas embodied in those items are of a kind which is readily assimilated by the child's existing cognitive structure. By contrast, a null or very small gain in performance level with no subsequent growth may suggest that the ideas involved are beyond the assimilative grasp of the child's existing cognition. The observation of some initial gain followed by a continued improvement over time, may indicate the actual formation of the structures required for the particular assimilative process to be carried out, i.e. may reflect the process of cognitive growth. For the first year group, such a process of conceptual change would have been that associated with the aspect of method-representation, the relatively small improvement in performance on the interpretation of letters items would, by this analysis, indicate that those children were not yet conceptually ready for the assimilation of this particular notion. By contrast, the second year children were able to demonstrate a ready assimilation of the representation of method ideas (as shown by the generally high gain on immediate posttest) which the younger children were in the process of acquiring, and showed that in their case the growth in cognition was being demonstrated with respect to the notion of letter as generalised number, which the younger children had generally appeared unable to acquire.

Alternatively, of course, the difference in effect of the programme with the two groups may have been due simply to a difference in emphasis of the teaching. As the experimenter taught both groups and noted no particular variation in approach in this regard, however, this explanation may be less likely, although it cannot be discounted on this basis alone.

<u>Performance by item: third and fourth year groups.</u> As in the case of the second year group, the level of pretest performance of the third year group was uniformly low across items (see Figures 8.4 and 8.5). However, whereas the second year delayed posttest results show a generally high level of performance across items, the same is not true for the third year group. In the latter case, whilst a notable improvement in delayed as opposed to pretest performance is discerned for the set of items associated with conjoining and with the use of

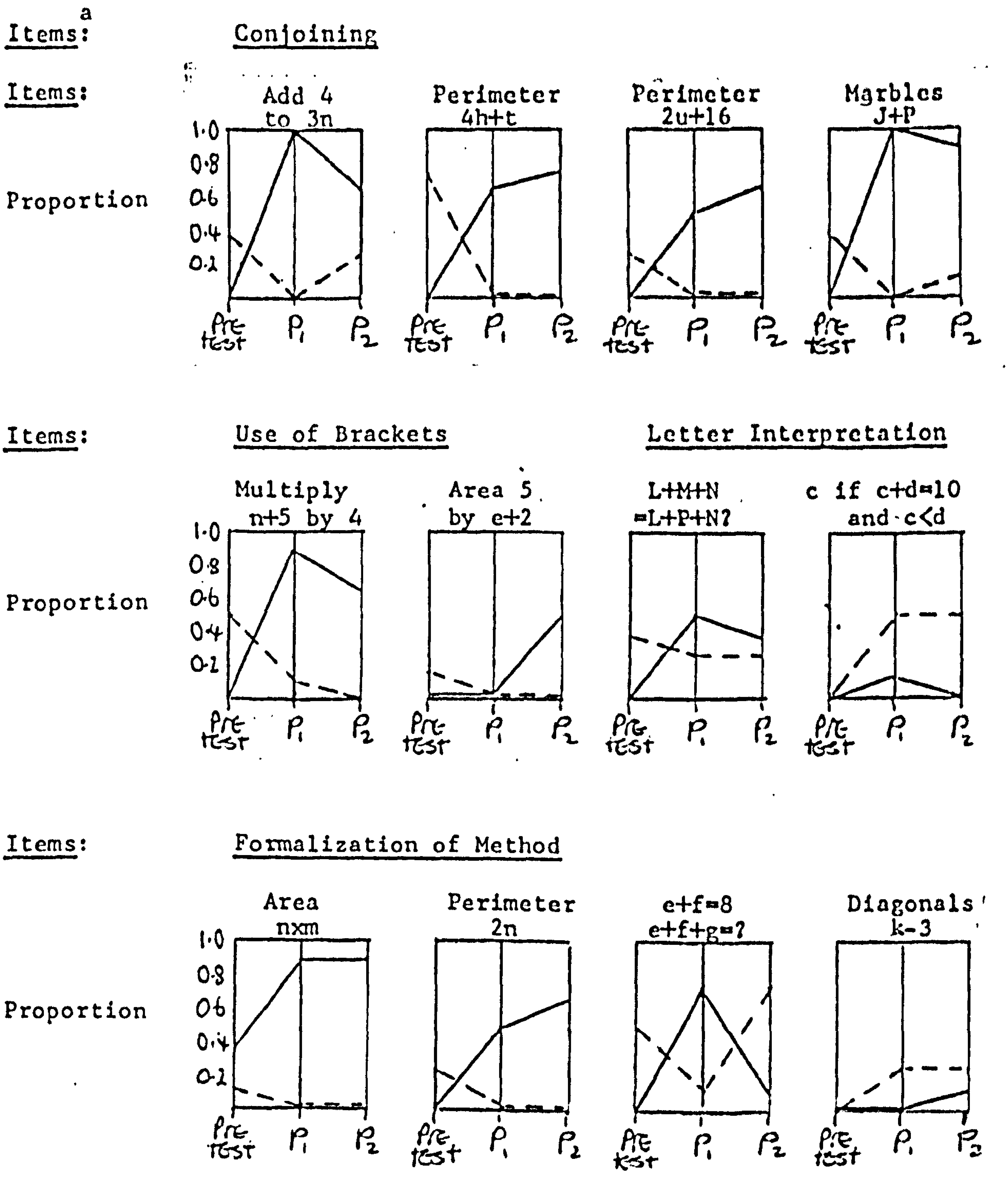


Figure 8.5 Proportion of children (n=8) giving correct (solid line) and 'error' (broken line) answers to individual items on pretest, immediate posttest (P_1) and delayed posttest (P_2) for 3rd year (age 14) class teaching (researcher) groups.

a Brief descriptions only of each item are given here. The full items appear in the pretest given in Appendix 6.

brackets, there is very little change in performance on those items considered to represent the interpretation of letters question, and those associated with formalization of method show rather mixed results. While three of the latter items show some considerable gain in performance (namely those asking the area of a rectangle measuring n by m, the perimeter of a figure with n sides each of length 2, and the area of a rectangle measuring 5 by e+2), two other items show more equivocal results. The '8+g' item ('if e+f=8, e+f+g=?') is interesting in that a substantial improvement in performance was observed between pre- and immediate posttests, but that this was followed by a marked drop in performance on the delayed posttest. Since this class had been taught the solution of simple equations (e.g. x+3=8) in the period between the immediate and delayed posttests, and since of all the test items used the '8xg' item most resembles an equation, it is tempting to suggest that the decline in performance on this item may have been due at least in part to a confusion in the children's minds between this item and the equations they had recently been solving. The pronounced increase in incidence of numerical answers to this item between the two posttests may provide support for this explanation. Certainly the decrease in performance on this item between immediate and delayed posttests is greater than for any other item and would seem to warrant some external explanation. The item requiring an expression for the number of diagonals in a polygon ('k-3') was also answered correctly by only one child on the delayed posttest. However, this item had the highest omission rate of the test, with five out of the eight children neglecting to provide a response. It may have been, therefore, that this item was simply not understood by the children, or that the children were discouraged by its perhaps unfamiliar nature.

The teaching programme may, therefore, have achieved some measure of success with this group in terms of an improved ability to formalize and symbolise method, as it appears to have done for the issues of use of brackets and conjoining. It would appear to have been less effective where the appreciation of letters as generalised number is concerned, although there is evidence of some gain in this regard.

A similar lack of improvement on the level of letter interpretation items was also noted for the fourth year group (see Figure 8.6). As indicated in the observations made on the actual teaching, this group was particularly resistant to the idea that a letter could represent a range of values rather than just one, and especially reluctant to accept that different letters could be assigned the same value. The obtained results therefore reinforce the observation that the teaching programme was less successful in restructuring students' conceptions in this regard. Apart from these items, the results for the fourth year group indicate that the programme was successful in encouraging an improved level of performance in each of the areas of conjoining, use of brackets, and representation of method.

Summary. In general, the difference in level of performance between pre- and posttests for each group on each set of items was sufficiently great to suggest that the teaching programme was effective in improving children's level of understanding (as measured by success on the items) in each of the areas of difficulty identified by the study with the exception of that relating to letter as generalised number. That success on the items does reflect 'understanding' is suggested to be demonstrated by the maintained or

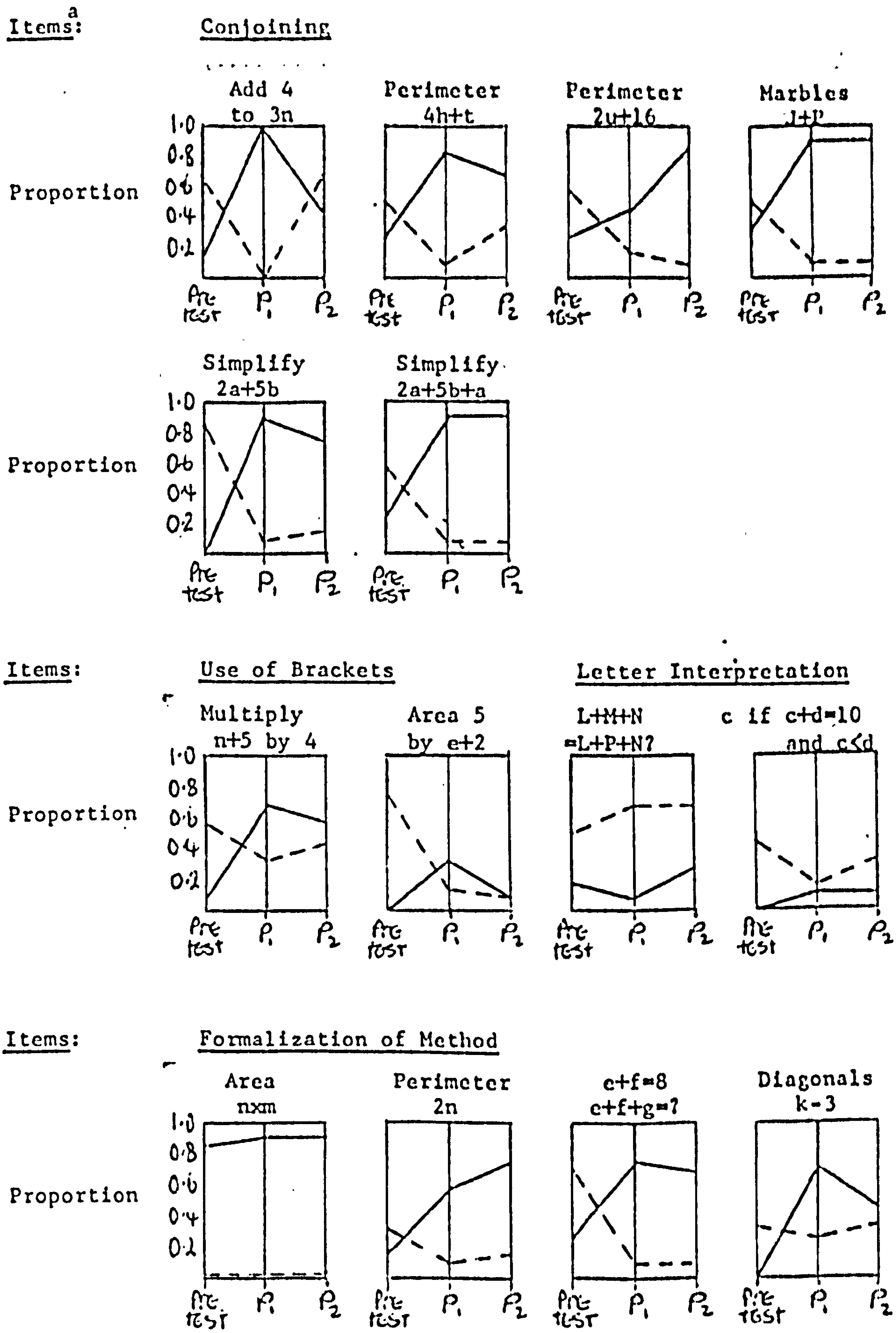


Figure 8.6 Proportion of children (n=12) giving correct (solid line) and 'error' (broken line) answers to individual items on pretest, immediate posttest (P_1) and delayed posttest (P_2) for 4th year (age 15) class teaching (researcher) group.

a. Brief descriptions only of each item are given here. The full items appear in the pretest given in Appendix 6.

improved level of performance over the two-month period intervening between immediate and delayed posttests. Nevertheless, the results also show that there is still room for improvement in selected areas for each group. The main area requiring improvement (for first, third and fourth year groups) is that concerning the level of letter interpretation. In addition, further work on the representation of method may have been beneficial for the first year group, and on the use of brackets for the second year group. These results thus bear out the observations made during the actual teaching.

Modification of the Teaching Module

The only modifications made to the teaching programme at this stage related to the material dealing with the use of brackets, and that relating to the algebraic representation of generalised method. In line with the observations made during the teaching and which have been discussed earlier, the work specifically dealing with the possible ambiguity of expressions in which brackets were not used was omitted, and the programme reverted to the earlier approach by which the conventions for the order of operations were overridden and the use of brackets was introduced as a machine requirement to specify the order in which operations were to be performed.

In addition, some small changes were made to the work on generalisation, namely by including a wider range of problems in terms of both content and difficulty. The final version of the teaching module as thus amended is given in Appendix 5. The pre- and posttests were also amended slightly to include an extra item on the use of brackets and on the formalization of method. These test amendments are described in Appendix 10.

The teaching module was assembled into a 'kit' comprising a detailed set of teaching notes, including a brief outline of the research findings and a rationale for the programme, and a class set of all worksheets to be used in the programme together with a set of 'answer sheets' on which all work done by the children would be recorded. This kit was then distributed to those teachers who had volunteered to participate in the next phase of the research.

Class Teaching (Other Teachers)

Sample

Seven teachers from five schools (designated as schools A-E) not previously used in the study agreed to participate in the class teaching (other teachers) phase of the research. A brief description of the schools and classes is given in Table 8.4.

Teacher Briefing

The teachers received varying degrees of briefing from a discussion of the research and its findings to a very cursory introduction to the management of the teaching programme alone. In all cases, however, the teachers were asked to:

a) follow the teaching notes and programme as closely as possible;
b) teach in a continuous fashion so that one component of the module followed another, without necessarily waiting for a new lesson in which to commence a new part of the programme;
c) make a note of any problems that were encountered;
d) make a note of any deviations from the programme, or any additional material or explanation used but only to deviate in this manner if it were deemed absolutely essential to the children's understanding;
e) collect all completed worksheets;
f) keep a record of their own observations on the programme as teaching progressed, and to note the amount of time (approximately) spent on each section;
g) keep a record of the children's attendance during the teaching programme.

Table 8.4

Class Teaching (Other Teachers) Sample

School	School Type (Location)	Year Group	Class and Description
A	Co-ed. Comprehensive (Hertfordshire)	1st Year (Age 12)	1a: Lower middle stream (band above remedial)
B	Co-ed. Comprehensive (Kent)	2nd Year (Age 13)	2a: top stream 2b: middle stream 2c: lower stream
C	Girls Comprehensive (North London)	3rd Year (Age 14)	Lower middle stream (lowest CSE group)
D	Co-ed. Comprehensive (North London)	4th Year (Age 15)	Mixed ability (CSE)
E	Co-ed. Comprehensive (North London)	4th Year (Age 15)	Middle stream (CSE)

Procedure

Three teachers (schools C, D and E) carried out the teaching and testing programme with one class each, of their own choosing. In schools A and B it was possible to involve other classes and so a departure from the class teaching (researcher) approach was adopted in these cases in order to introduce an element of 'controlled' comparison. The teacher in school A arranged for another teacher to teach introductory algebra using the textbooks and approach normally used in that school (see Appendix 11 for details) to a parallel class, and administered pre- and posttests to both groups of children. In the case of school B, two more teachers in the same school were recruited to teach the programme, so that one class from each of the three ability bands in the school could be given the teaching module material, while a parallel class in each band was given the pre- and delayed posttests in order to ascertain that any observed improvement in performance on the part of the taught classes could not be attributed simply to maturation. In fact, the results from the CSMS longitudinal study had revealed relatively little improvement in performance in terms of level attained over a two year period (Küchemann, 1980; Hart, 1980b; see also Table 4.5), so that it was not expected that large gains would be observed over the three to four month period intervening between pretest and delayed posttest in the present study. Nevertheless, it was thought useful to investigate this issue in a more controlled setting since the opportunity presented itself. The design of the class trial programme is summarised in Table 8.5.

Results

School A. Test performances in terms of total score obtained on

Table 8.5

Design of Class Teaching (Other Teachers) Programme

School	Class	Treatment
A	Experimental	Pretest/Teaching Module/Immediate and Delayed Posttests
	Control	Pretest/Standard Algebra Teaching/Immediate and Delayed Posttests
B	Experimental	Pretest/Teaching Module/Immediate and Delayed Posttests
	Control	Pretest/No Algebra Teaching/Delayed Posttest only
C,D,E	Experimental	Pretest/Teaching Module/Immediate and Delayed Posttests

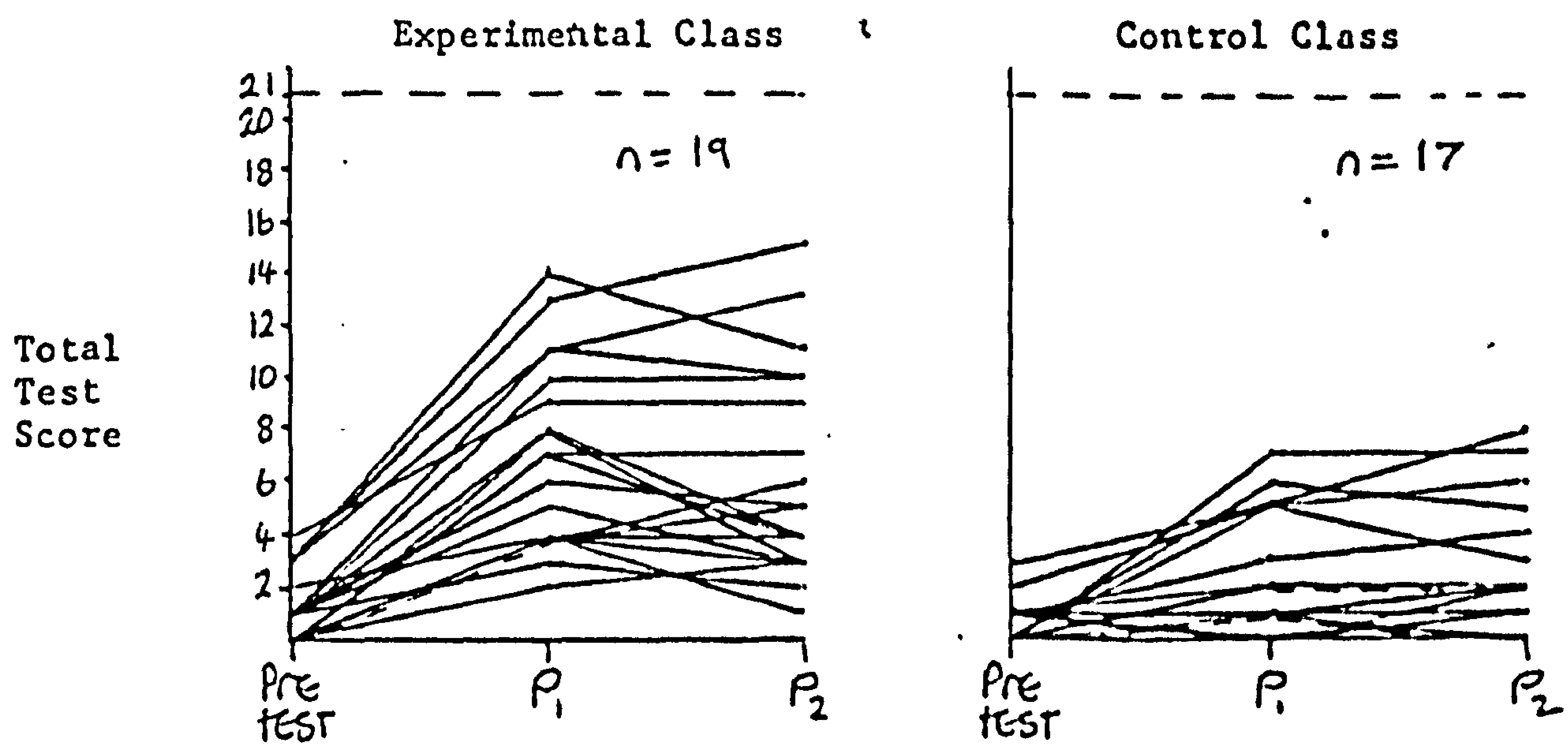


Figure 8.7 Total test score obtained on the pretest, immediate posttest (P_1) and delayed posttest (P_2) by individual children in the 1st year (age 12) experimental and control classes (school A) in the class teaching (other teachers) study.

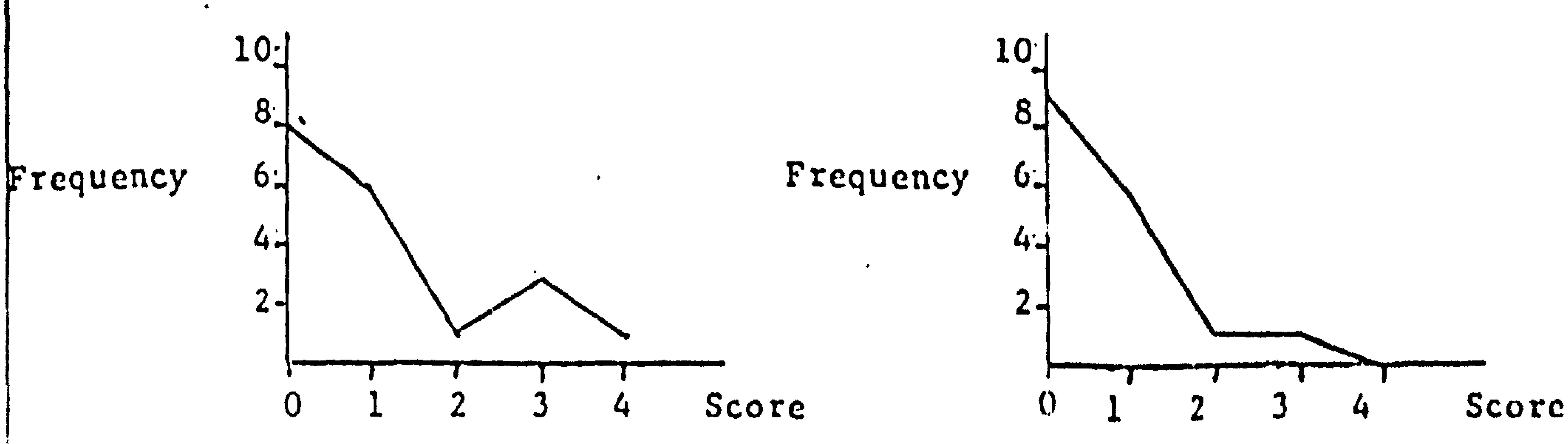
Table 8.6

Comparison of Pre- and Posttest Performance for 1st Year (Age 12) Experimental and Control[a] Classes (School A)

Pretest vs Immediate Posttest (Maximum Score = 21)								
Class	Pretest Mean	Posttest Mean	Mean Gain	s.d.	N	t	df	p
Experimental	1.1	7.1	5.94	2.86	19	8.81	18	<.001
Control	0.7	2.4	1.70	2.29	17	2.97	16	<.005
Pretest vs Delayed Posttest (Maximum Score = 21)								
Experimental	1.1	6.2	5.10	3.31	19	6.54	18	<.001
Control	0.7	2.7	2.00	2.25	17	3.56	16	<.005

Experimental vs Control Gains: Immediate Posttest vs Pretest						
Class	Mean Gain (Immediate Posttest) (vs Pretest)	s.d.	N	t	df	p
Experimental	5.94	2.86	19	4.80	34	<.001
Control	1.70	2.29	17			

Experimental vs Control: Comparison of Pretest Performance



a. The control class was taught introductory algebra using the textbooks and procedure usually used in that school (see Appendix 11 for details). This teaching occurred between pre- and immediate posttests. The content of the control class programme was essentially similar to that of the experimental teaching module, so that the test items were as appropriate to the control group as to the experimental group.

No algebra teaching was covered for either group between the immediate and delayed posttests.

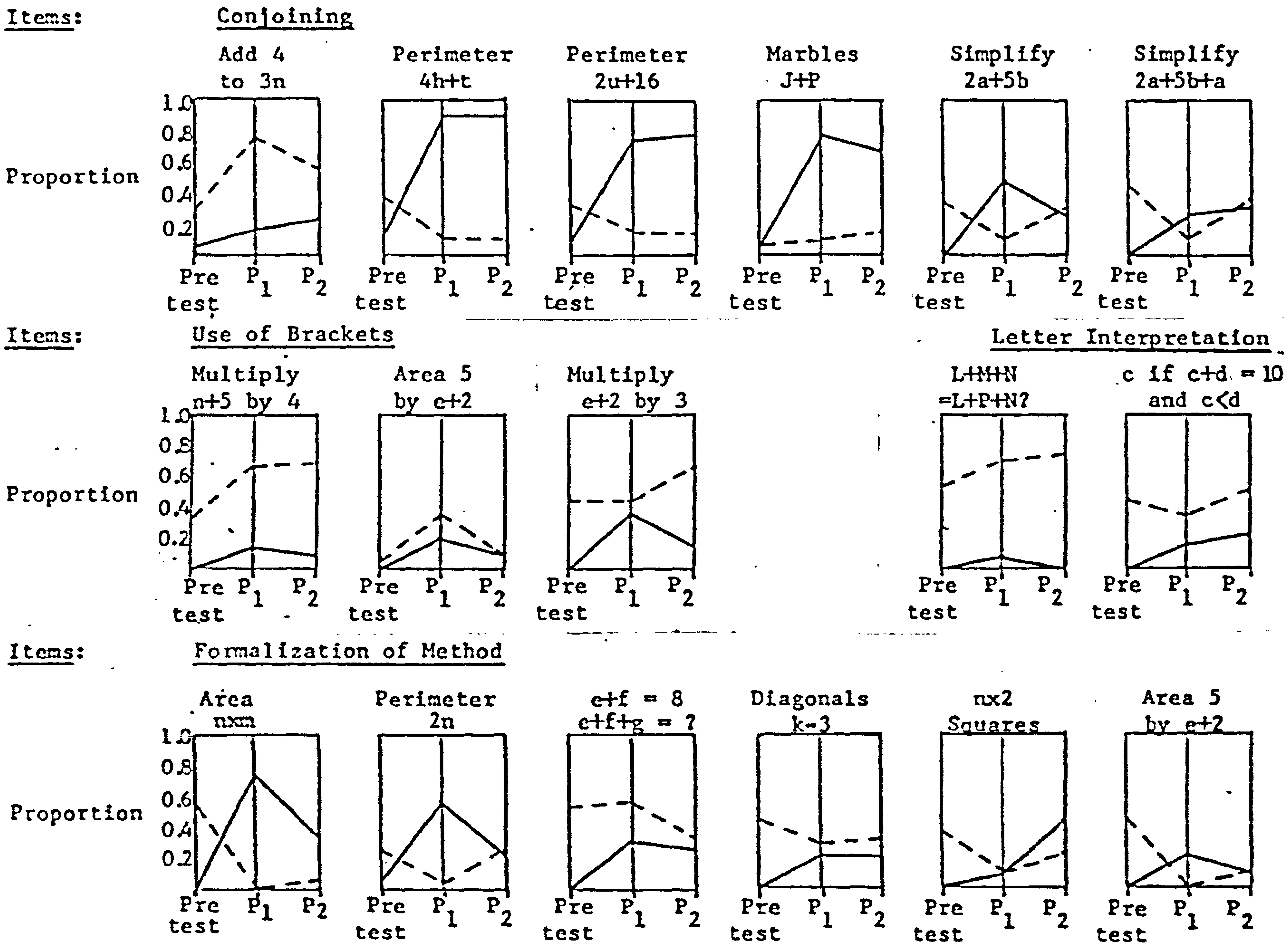


Figure 8.8 Proportion of children (n=19) giving correct (solid line) and 'error' (broken line) answers to individual items on pretest, immediate posttest (P_1) and delayed posttest (P_2) for 1st year (age 12) experimental class (school A) in the class teaching[1] (other teachers) study.

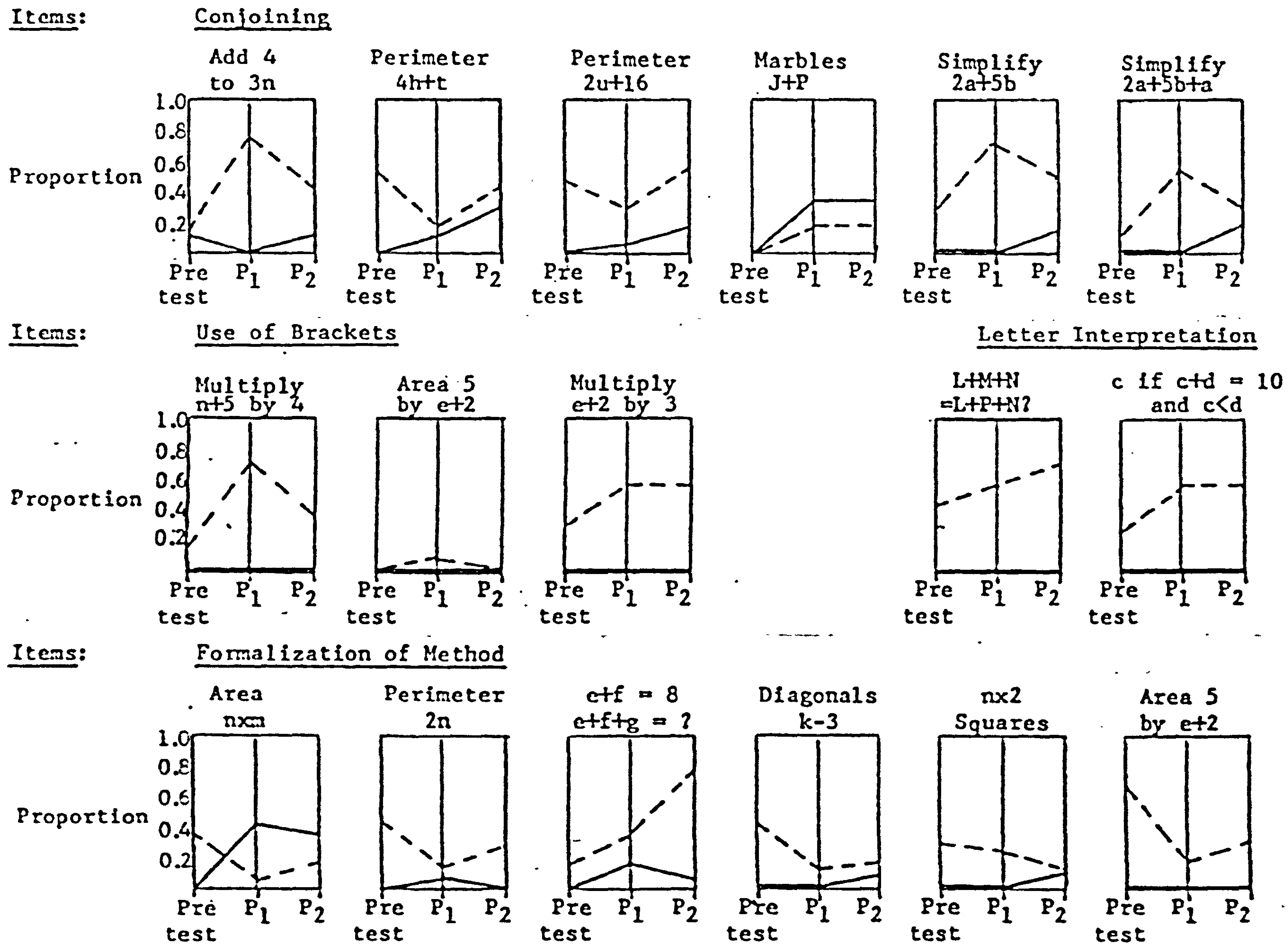


Figure 8.9 Proportion of children (n=17) giving correct (solid line) and 'error' (broken line) answers to individual items on pretest, immediate posttest (P_1) and delayed posttest (P_2) for 1st year (age 12) control class (school A) in the class teaching (other teachers) study.

algebra (see Table 8.7). This may be regarded as supporting the suggestion that conjoining is the result of a more fundamental process or conception, and is not, for example, the result of a confusion with the abbreviated algebraic product consequent upon the teaching of the latter notation.

Also high was the pretest level of 'letter as specific unknown' responses to the level of letter interpretation items (Figures 8.8 and 8.9). This suggests that children may have a 'natural' tendency to view a letter as standing for one value (when the letter is viewed as representing number rather than something else), and to regard different letters as representing different values. In the absence of specific teaching to the contrary, it would perhaps not be surprising if children adhered to these ideas.

Whilst the teaching programme appeared not to have been effective in assisting children to reassess their interpretation of letters in this way, it did seem to have been useful in promoting a positive attitude amongst children in this group towards letters and their use. This is indicated by the teacher's comments on the programme, for example:

> "One of the most beneficial results of the work has been the positive attitude of the children towards this topic. They have developed an enthusiasm for algebra and using letters which I have found difficult to get using the usual text books and worksheets.
> Particularly striking is the module's encouragement of the child's understanding of the subject, as opposed to rote learning of techniques or the mere production of answers .... Instead of searching for the right arithmetic answers, which is not a strong point (of this class), the children

Table 8.7

Proportion of Children in Each Year Group
Giving Indicated Error Answer to Selected Pretest Items

Item	1st Yr (Middle)	2nd Yr (Top/Middle/Lower)			3rd Yr (Middle)	4th Yr (Middle)
	Year Group (Ability Band)					
<u>Conjoining</u>	(n=19)	(n=27)	(n=20)	(n=15)	(n=16)	(n=25)
Add 3 to 4n	.32	.37	.60	.53	.19	.42
Perimeter 4h+t	.37	.00	.55	.47	.31	.17
Perimeter 2u+16	.32	.30	.75	.53	.38	.50
Simplify 2a+5b	.37	.33	.70	.60	.31	.22
<u>Letter as Specific Unknown</u>						
L+M+N = L+P+N?	.53	.56	.65	.53	.44	.65
c+d = 10, c<d	.42	.48	.50	.87	.56	.42

were being encouraged to explore what they were doing and the methods they were using .... This focus is particularly important in teaching mathematics to children at this ability level, as there is a real danger of mathematics teaching becoming, for them, a process of indoctrination into meaningless procedures and techniques. The approach of the module enabled the children to achieve more success with a greater feeling of control."

The teacher also commented upon one or two useful outcomes of particular sections of the work, for example:

"In discussing (an example concerning with writing the instructions to find the perimter of 'any rectangle'), one boy noticed that $2x + 2y$ could be expressed as $2(x+y)$, which I found surprising and most rewarding. This led the class to the observation that $2n + 2 = 2(n+1)$."

There were, however, several criticisms of the material, including that concerning the level of language used (some of the work-sheets use words such as 'multiply' and 'divide' whereas 'times' and 'goes into' are more readily understood by many children). In addition, one or two of the problems are somewhat wordy, and furthermore are rather artificial. There is a real need to supplement the material with many more simpler and more 'meaningful' problems, and this weakness has been commented upon by other teachers, as well as being recognised by the researcher.

School B. Individual changes in total score over the pre- and posttests for each of the three experimental classes are shown in Figure 8.10.

The gains in total score obtained between pretest and immediate posttest averaged 4, 6.9 and 4.3 for the top, middle and lower ability

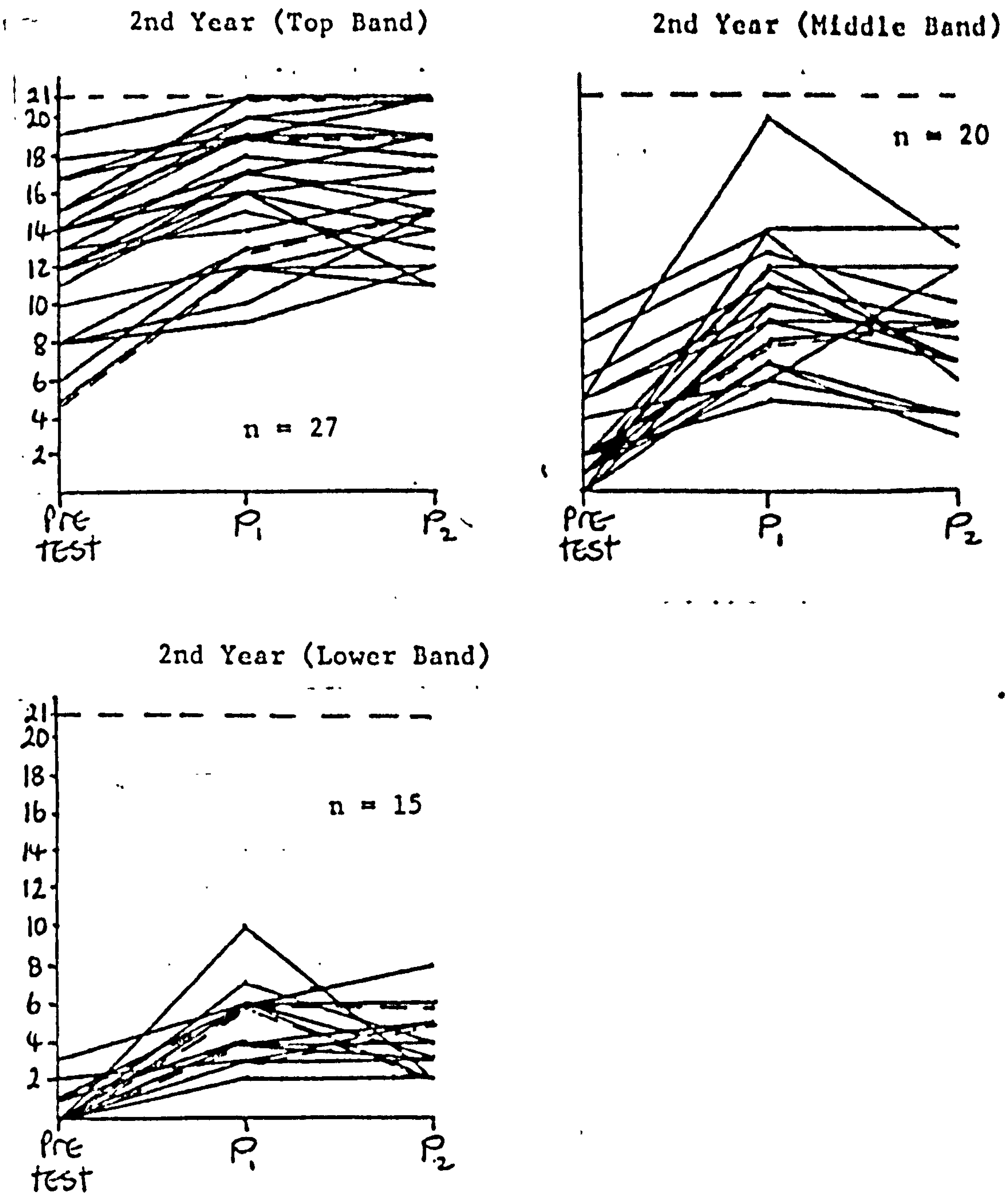


Figure 8.10 Total test score obtained on the pretest, immediate posttest (P_1) and delayed posttest (P_2) by individual children in the three 2nd year (age 13) experimental classes (school B) in the class teaching (other teachers) study.

Table 8.8

Comparison of Pre- and Posttest Performance for 2nd Year (Age 13) Experimental and Control[a] Classes (School B)

Ability Band	Pretest Mean	Posttest Mean	Mean Gain	s.d.	N	t	df	p
Pretest vs Immediate Posttest (Experimental Groups) (Maximum Score = 21)								
Top	12.4	16.4	3.96	1.95	27	10.35	26	<.001
Middle	2.9	9.8	6.90	3.61	20	8.33	19	<.001
Lower	0.7	5.0	4.27	2.22	15	7.20	14	<.001
Pretest vs Delayed Posttest (Experimental Groups) (Maximum Score = 21)								
Top	12.4	16.5	4.04	2.23	27	9.24	26	<.001
Middle	2.9	7.8	4.85	3.07	20	6.89	19	<.001
Lower	0.7	3.7	3.00	1.51	15	7.43	14	<.001
Pretest vs Delayed Posttest ('Control' Groups) (Maximum Score = 21)								
Top	14.7	15.1	0.40	2.37	25	0.83	24	NS
Middle	6.3	10.1	3.75	3.09	24	5.82	23	<.001
Lower	1.6	2.1	0.55	2.04	20	1.18	19	NS

a. The control classes received no algebra teaching during the period of investigation from pre- to delayed posttests, except for the middle ability band control class who were taught this topic immediately prior to the delayed posttest (see text). The control classes received only the pre- and delayed posttests, in an attempt to monitor any changes in performance due to maturation.

classes respectively (see Table 8.8). The average gain for the top ability class between pretest and delayed posttest was 4 and for the lower ability class was 3, as compared with the respective control classes' mean gains of 0.4 and 0.6 for these two ability groups. These latter figures provide support for the view that the gains observed for the experimental classes were not due simply to maturation. The picture is, however, clouded by the results for the middle ability classes, where the pretest to delayed posttest mean gain was 4.9 for the experimental group and 3.8 for the control group. However, in this case the teacher of the control class did not administer the delayed posttest until three weeks after the experimental class had received the test, and was in fact teaching algebra to the control group during this time. This effectively makes the test an immediate posttest following an alternative teaching programme, so that gains in performance would be expected for this particular group. From this point of view, the resulting mean gain of 3.8 is perhaps more meaningfully compared with the mean gain of 6.9 between pretest and immediate posttest for the experimental class.

The results for individual items for each of the three experimental classes are shown in Figures 8.11, 8.12 and 8.13. For the top ability band class, an improvement in performance as measured by an increase in the incidence of correct answers and a decrease in the frequency of target errors, was apparent for each of the areas of difficulty under study (see Figure 8.11). With the exception of the items relating to the use of brackets, this improvement was maintained on the delayed posttest. In the case of the use of brackets items, the marked drop in performance between immediate and delayed posttests indicates that the ideas relating to this aspect were generally not assimilated by the children. Possibly the restriction of this aspect

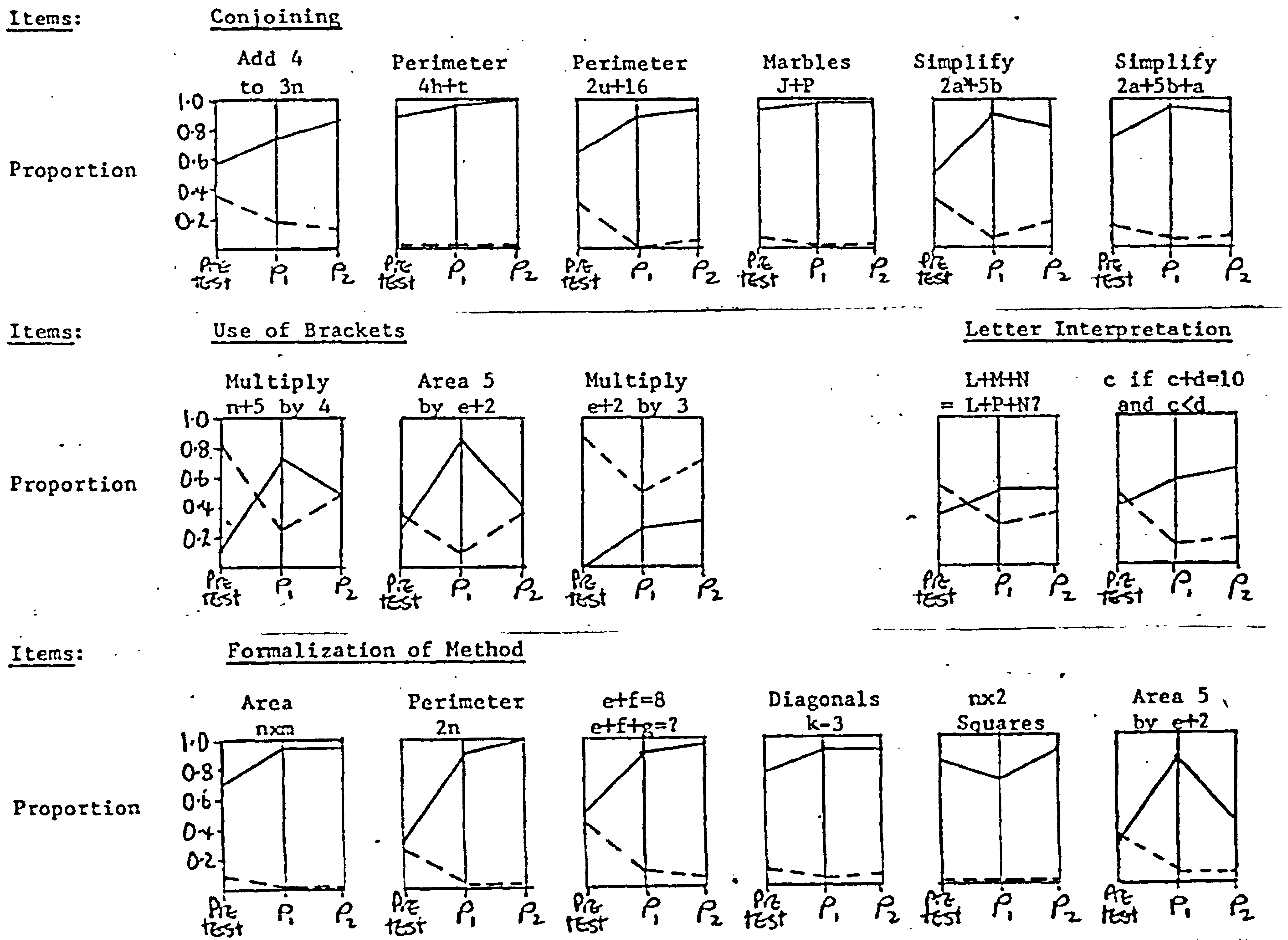


Figure 8.11 Proportion of children (n=27) giving correct (solid line) and 'error' (broken line) answers to individual items on pretest, immediate posttest (P_1) and delayed posttest (P_2) for 2nd year (age 13) top band experimental class (school B) in class teaching (other teachers) study.

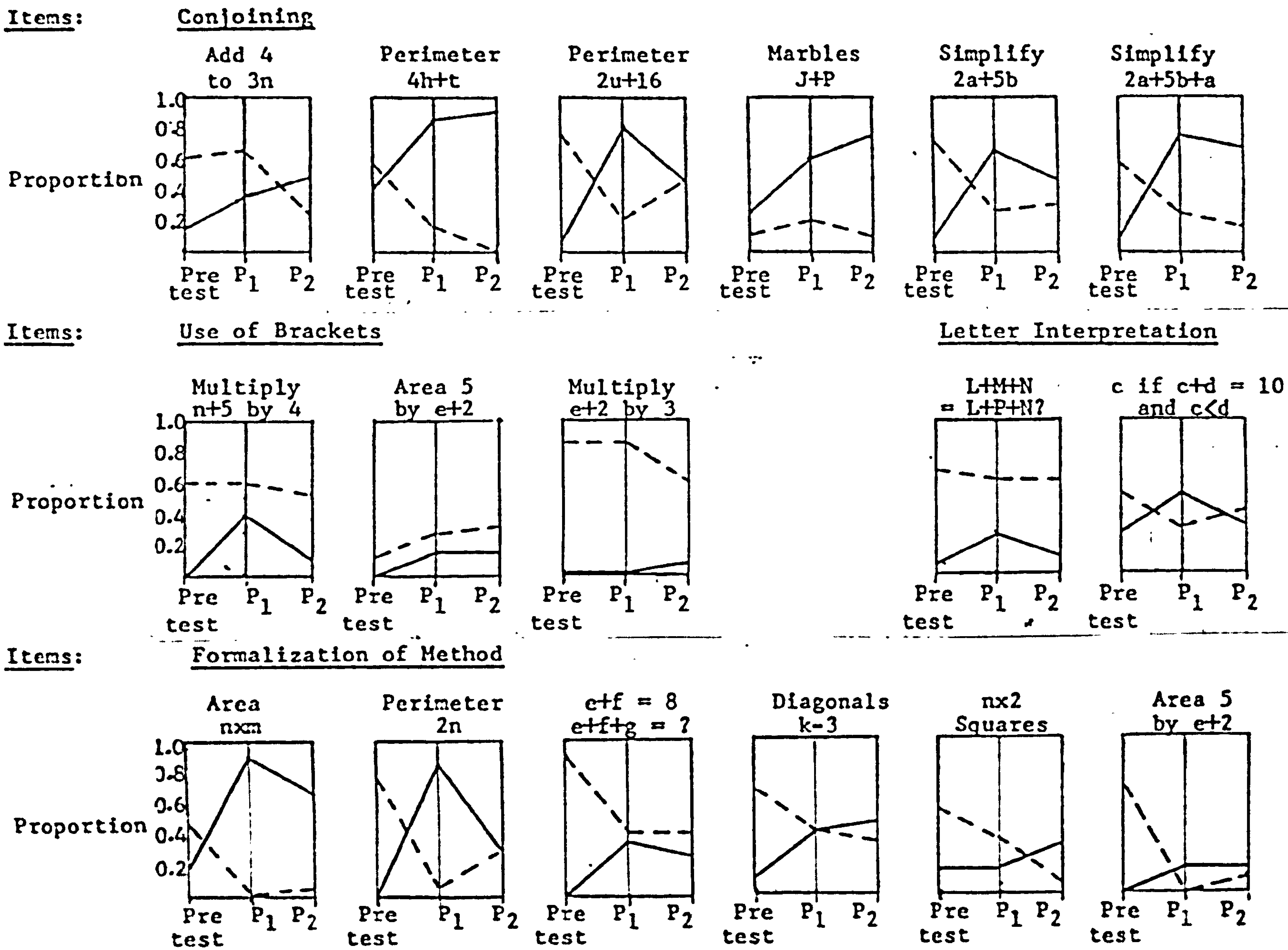


Figure 8.12 Proportion of children (n=20) giving correct (solid line) and 'error' (broken line) answers to individual items on pretest, immediate posttest (P_1) and delayed posttest (P_2) for 2nd year (age 13) middle band experimental class (school B) in class teaching (other teachers) study.

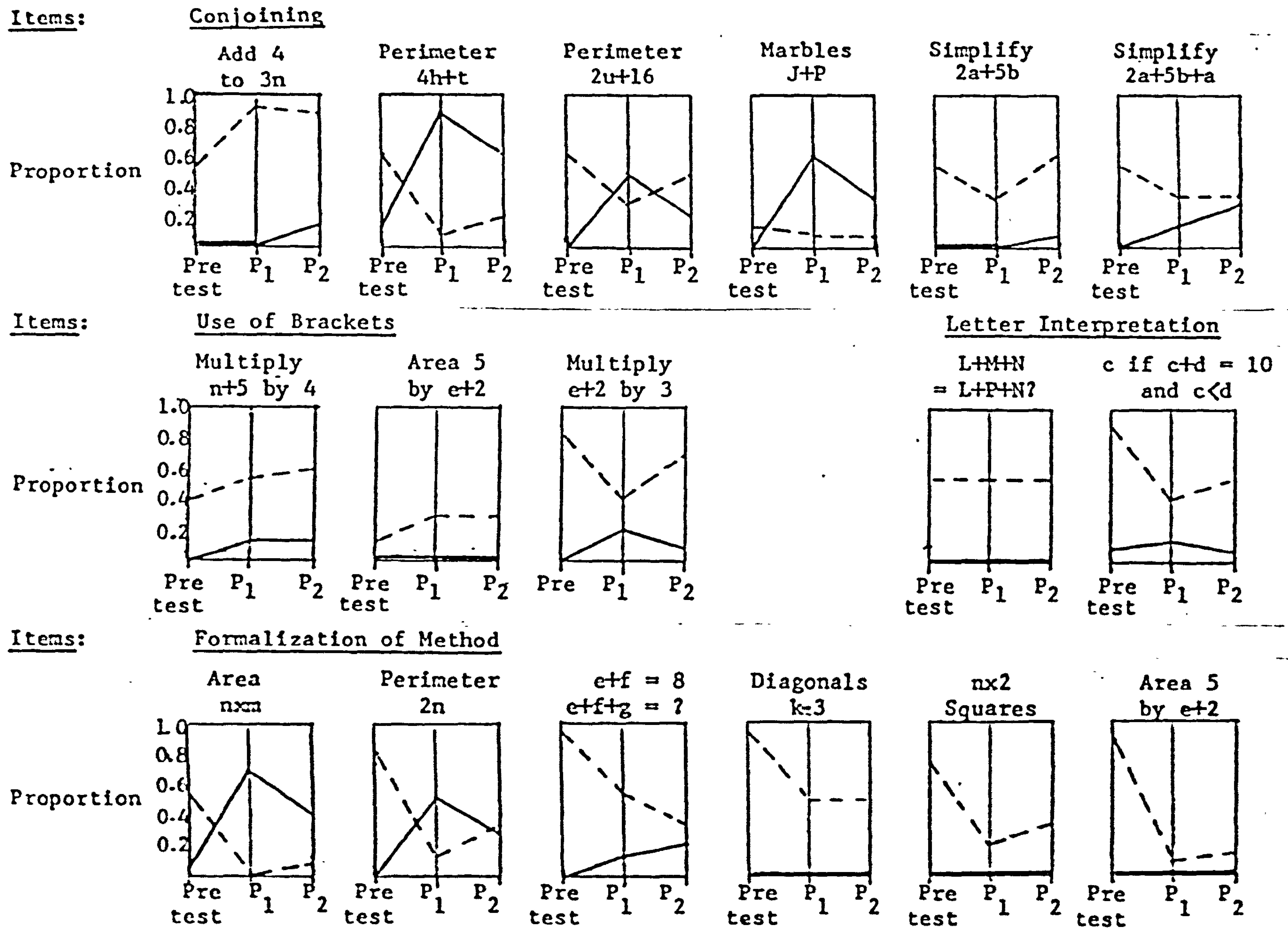


Figure 8.13 Proportion of children (n=15) giving correct (solid line) and 'error' (broken line) answers to individual items on pretest, immediate posttest (P_1) and delayed posttest (P_2) for 2nd year (age 13) lower band experimental class (school B) in class teaching (other teachers) study.

to the mathematics machine context may be partly responsible for this occurrence.

This difficulty over the use of brackets, already highlighted in the earlier teaching studies, tests and interviews, was indicated again in the data for the middle and lower ability mathematics groups (see Figures 8.12 and 8.13). These show little or no overall increase in facility for the use of brackets items, and either no change or an actual increase in the frequency of errors (in terms of bracket omissions). However, the items for which this increase in error occurred (namely the 'area of a rectangle measuring 5 by e+2' item for the middle ability group, see Figure 8.12, and both this and the 'multiply n+5 by 4' item for the lower group, see Figure 8.13), were also marked by a substantial decrease in the number of children giving a purely numerical answer. Since the omission of brackets is rarely detectable in a numerical answer, it is likely that the increase in bracket-omission errors was noted precisely because children had moved from a numerical to an algebraic answer. Nevertheless, the programme was evidently relatively ineffective, as far as these children were concerned, in its attempt to structure their thinking with respect to the use of brackets.

Both middle and lower mathematics groups appeared to have made gains on the items in which conjoining is a potential error, though these gains were substantially greater for the middle group than for the lower. Also notable was the general improvement for both groups on the formalization of method items, which were marked by a pronounced decrease in the frequency of related error, although the increase in correct responses was not great, particularly for the lower ability group. This would seem to suggest that while these children had come to appreciate that a numerical or alphabetic answer was not appropriate to the questions concerned, they were as yet still unable to produce the correct algebraic expression. Further work on the representation of method in arithmetic as well as algebra would probably be beneficial to these children. As in the case of several of the other groups of children, the issue relating to letters as generalised number was not well handled by the programme in the case of these children.

Schools C,D,E. Table 8.9 and Figure 8.14 show the changes in total score on the pre- and posttests for the three classes from schools C, D and E. In each case, mean performance on the immediate posttest was significantly higher than the mean pretest performance. For all three classes there was a decrease in performance between immediate and delayed posttests, but the mean delayed posttest score was still significantly higher than that of the pretest, showing that there had been some overall improvement of a more lasting kind.

The results for change in performance on individual items over the three tests are giving in Appendix 12. The data generally support the findings from the other class trials in showing a greater effectiveness of the programme where the foregoing of conjoining in algebraic addition (or the acceptance of unclosed answers) is concerned, and also for the representation of method aspect. While an initial increase in facility for the use of brackets items was observed for the fourth year classes in particular, this was generally not maintained on the delayed posttest, and the proportion of bracket-omission errors remained high on all three tests. Results for the fourth year groups on the interpretation of letters items were also equivocal, in that while there was evidence of a maintained increase in facility on these items following the teaching programme, and the error incidence showed an initial decrease, there was some recovery in

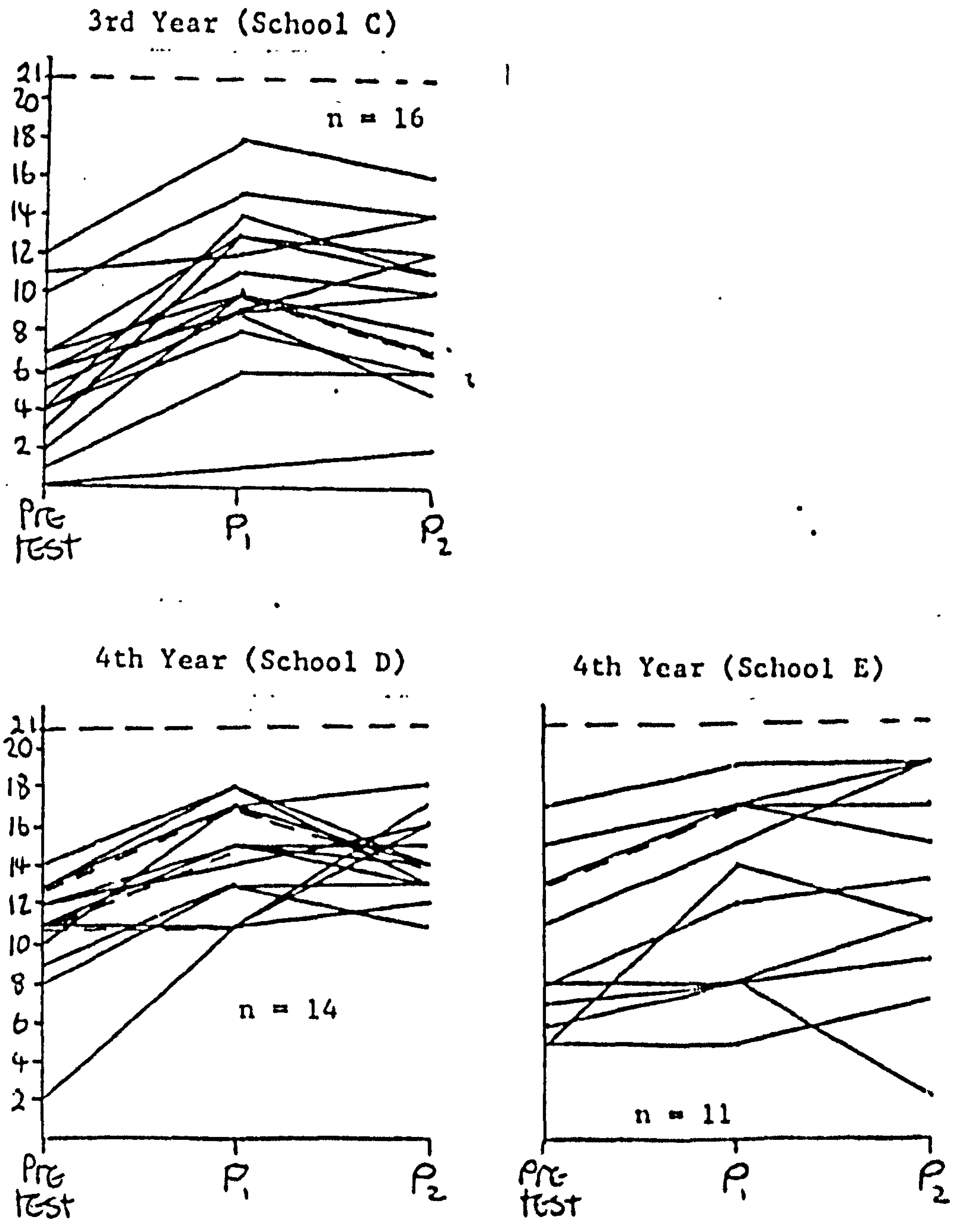


Figure 8.14 Total test score obtained on the pretest, immediate posttest (P_1) and delayed posttest (P_2) by individual children in the 3rd year (age 14) and 4th year (age 15) experimental classes (schools C, D and E) in the class teaching (other teachers) study.

Table 8.9

Comparison of Pre- and Posttest Performance for
3rd Year (Age 14) and 4th Year (Age 15)
Experimental Classes (Schools C, D and E)

Pretest vs Immediate Posttest									
School	Year Group (Age)	Pretest Mean	Posttest Mean	Mean Gain	s.d.	N	t	df	p
C	3rd Year (Age 14)	5.5	10.5	4.94	2.59	16	7.39	15	<.001
D	4th Year (Age 15)	10.7	14.6	3.93	2.37	14	5.98	13	<.001
E	4th Year (Age 15)	9.8	12.7	2.91	2.55	11	3.61	10	<.005
Pretest vs Delayed Posttest									
C	3rd Year (Age 14)	5.5	9.4	3.94	2.33	16	6.55	15	<.001
D	4th Year (Age 15)	10.7	14.3	3.57	3.81	14	3.38	13	<.005
E	4th Year (Age 15)	9.8	12.9	3.09	2.91	11	3.36	10	<.005

the frequency of error between the immediate and delayed posttests, and the level of error remained relatively high.

A General Note

On average, the gains in performance resulting from the class teaching programme were in the order of 3 to 7 items correct out of a total of 21. During the class teaching (researcher) phase the mean gains had been in the order of 8 to 12 (out of 21 items) and 7 (out of 16 items). The results are not comparable, however, since:

(a) in the latter case the children had been specifically chosen because they were making the identified errors, whereas in the 'other teacher' phase the groups were intact classes and so would have contained a proportion of children who were not making the errors, and to whom the teaching programme would therefore have been less appropriate;

(b) pretest levels of performance were higher in the 'other teacher' phase, even when the subgroups of children identified as making the errors in question are considered apart from the intact classes (see Table 8.10).

A partition of the 'other teacher' whole class groups into 'error' and 'non-error' types respectively, reveals a higher mean gain on the part of the former children as might be expected in the light of the comment made in point (a) above (see Table 8.11 and Figure 8.15). The classification of children as 'error' types (or not) was made for this purpose on the basis of the same criteria used previously in the 'researcher' taught phase of the study (given in Chapter 5). The separation of results in this way creates a set of subgroups (the 'error' groups) for the 'other teacher' phase which are now more akin to the groups used in the researcher-taught phase, but

Table 8.10

Pre- and Posttest Mean Performance of 'Error' Subgroups in Researcher Taught Phase and Class Teaching (Other Teachers) Phase

Year Group	Researcher				Other Teachers			
	Pretest	P_1	P_2	n	Pretest	P_1	P_2	n
1st (Age 12)	-	10.0	12.0	14	1.1	7.1	6.2	13
2nd (Age 13)	3.5	14.4	15.4	16	10.2[a]	14.8	15.6	14
3rd (Age 14)	1.0	8.6	8.0	8	5.0	11.1	10.0	11
4th (Age 15)	5.2	13.1	12.6	12	9.3 8.0	14.1 12.3	14.6 13.5	8 4

a. 2nd year data are for the top ability band, as this group was most comparable to the top stream 2nd year group used in the class teaching (researcher) phase

Table 8.11

Mean Gains in Performance of 'Error' and 'Non-Error' Subgroups Between Pre- and Posttests in Class Teaching (Other Teachers) Study

Year Group/ Level	Subgroup					
	'Error'[a]			'Non-Error'[b]		
	Pre-P_1	Pre-P_2	n	Pre-P_1	Pre-P_2	n
1st (Age 12)	6.0	5.1	13	3.0	2.2	6
2nd (Age 13)						
- Top	4.6	5.4	14	2.8	2.2	13
- Middle	7.7	5.6	16	5.0	3.5	2
- Lower	4.3	3.0	15	-	-	-
3rd (Age 14)	6.1	5.0	11	3.0	3.0	2
4th (Age 15)	4.8	5.3	8	3.2	1.0	6
	4.3	5.5	4	3.0	3.0	4

a. For criteria for children identified as making errors under study, see Chapter 5.

b. Data exclude results for children included in the whole class analysis (Tables 8.6 to 8.9) but who were absent for half or more of the teaching programme lessons.

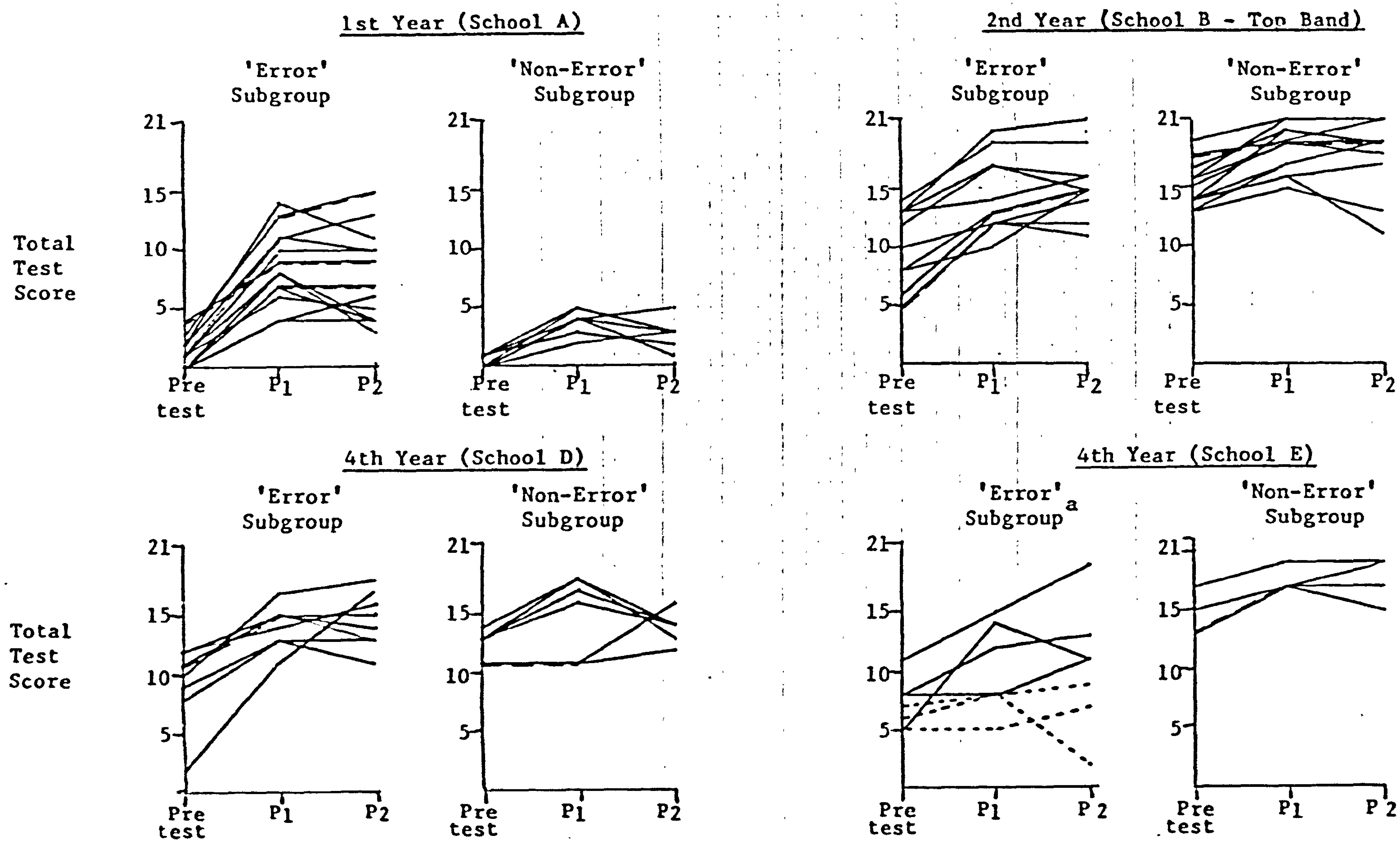


Figure 8.15 Total test score obtained on pretest, immediate posttest (P_1) and delayed posttest (P_2) by individual children in the class teaching (other teachers) study: A comparison of 'error' and 'non-error' subgroups.

a. Dotted line shows results for children who were absent for half or more of the teaching programme.

the problem of non-comparability in terms of different pretest levels of performance still remains. A reanalysis of the data showing change in performance over pre- and posttests on individual items for the 'error' groups only reveals a general pattern which is essentially similar to that already presented for the whole class data (Figures 8.8, 8.11 to 8.13 and Appendix 12) so that this reanalysis has not been illustrated here.

The separation of results in this way also permits consideration of the overall effect of the teaching programme on those children in the class-groups who were not making the errors under study. On average, it can be seen from Table 8.11 that those children's performance was not adversely affected by the teaching programme, but indeed that there was a general improvement in mean test score, albeit slight, for this group of children as well. This is supported by an analysis of the number of errors made by individual children in each error category (namely 'conjoining', 'use of brackets', 'letter interpretation' and 'formalisation of method') on the pre- and delayed posttests (Table 8.12). This analysis indicates a tendency on the part of all children to make fewer errors on the delayed posttest than on the pretest. The analysis also gives general support to the remarks made earlier concerning the differential effectiveness of the teaching programme with regard to the various areas of difficulty addressed.

A Note on 'Levels of Understanding'

Whilst the teaching experiment described here was not addressed specifically to increasing children's 'level of understanding' in algebra in the sense defined by the CSMS research, it was nevertheless considered that any improvement in children's performance brought

Table 8.12

Number of Errors in Each Error Category Made by Individual Children on Pre- and Delayed Posttests in Class Teaching (Other Teachers) Phase

Error Category

1st Year (Age 12)

Conjoining (6 items) — Pretest (rows) by Posttest (columns)

No. of Errors	6	5	4	3	2	1	0
6							
5							
4	0	0	1	1	1	0	1
3	0	0	0	1	2	1	0
2	0	1	0	0	0	3	0
1					0	1	1
0					2	1	1

Use of Brackets (2 items) — Pretest (rows) by Posttest (columns)

No. of Errors	2	1	0
2	0	0	0
1	1	6	2
0	0	7	3

Letter Interpretation (2 items) — Pretest (rows) by Posttest (columns)

No. of Errors	2	1	0
2	2	4	0
1	2	2	2
0	2	5	0

Formalization of Method (6 items) — Pretest (rows) by Posttest (columns)

No. of Errors	6	5	4	3	2	1	0
6					1	0	0
5					1	0	0
4					0	1	0
3					4	3	0
2					1	1	1
1		1	0	0	2	2	1
0							

2nd Year (Age 13)

Conjoining (6 items) — Pretest (rows) by Posttest (columns)

No. of Errors	6	5	4	3	2	1	0
6							
5						1	0
4					3	2	0
3				2	2	1	5
2				1	1	7	4
1				1	2	5	11
0				1	1	1	11

Use of Brackets (2 items) — Pretest (rows) by Posttest (columns)

No. of Errors	2	1	0
2	4	9	0
1	7	17	16
0	2	4	3

Letter Interpretation (2 items) — Pretest (rows) by Posttest (columns)

No. of Errors	2	1	0
2	8	12	7
1	5	10	7
0	1	3	9

Formalization of Method (6 items) — Pretest (rows) by Posttest (columns)

No. of Errors	6	5	4	3	2	1	0
6				2	3	2	1
5		1	1	0	6	2	0
4					2	4	4
3					1	2	5
2						2	3
1							12
0							9

(Table 8.12 Cont'd.)

Table 8.12 (Cont'd.)

Error Category

Conjoining (6 items)

	No. of Errors (Pretest)	Posttest 6	5	4	3	2	1	0
3rd Year (Age 14)	6							
	5							
	4					1	0	0
	3					1	1	1
	2					0	1	1
	1					3	4	0
	0						2	1
4th Year (Age 15)	6							
	5						1	1
	4						0	0
	3				1	1	1	0
	2				1	0	1	1
	1				1	1	4	4
	0					1	0	3

Use of Brackets (2 items)

	No. of Errors (Pretest)	Posttest 2	1	0
3rd Year (Age 14)	2	2	0	1
	1		8	1
	0		3	1
4th Year (Age 15)	2	8	5	1
	1		4	1
	0		3	0

Letter Interpretation (2 Items)

	No. of Errors (Pretest)	Posttest 2	1	0
3rd Year (Age 14)	2	5	3	0
	1	1	1	2
	0		4	0
4th Year (Age 15)	2	6	7	1
	1		6	2
	0			

Formalization of Method (6 items)

	No. of Errors (Pretest)	Posttest 6	5	4	3	2	1	0
3rd Year (Age 14)	6							
	5						1	0
	4					1	1	0
	3				2	1	2	0
	2				1	0	1	1
	1				0	0	2	3
	0							
4th Year (Age 15)	6					1	0	0
	5					1	0	1
	4			1	0	1	3	0
	3						2	2
	2						1	4
	1						2	1
	0							2

about as a result of the teaching programme might be reflected in a movement across 'levels' (as defined by CSMS). The use of an abridged and modified version of the CSMS algebra test (Appendix 10) as pretest and delayed posttest enabled levels 0 to 3 to be assigned to individual children, but not level 4. A comparison of the levels thus assigned to each child on the pre- and delayed posttests shows that of the 122 children involved in the 'other teachers' phase of the teaching experiment, a total of 60 children increased their level of 'understanding' as defined by CSMS by one or more levels, while a further 61 children maintained the same level, in general greatly consolidating their position at that level. Table 8.13 gives the percentage of children maintaining, or improving their level of understanding in this way, for each of the four year-groups used in the 'other teachers' study. As a basis against which to consider the data presented in Table 8.13, attention might be given to the percentage of second year (age 13) children maintaining or improving their level of understanding over the two-year period monitored by the CSMS longitudinal study, as given in Chapter 4 (Table 4.5).

Summary

In general, the results of the class trials suggest that the teaching programme developed on the basis of the information obtained from the clinical interviews was effective in improving children's general level of understanding in elementary algebra, as measured by a sustained improvement in performance on test items in this topic. As far as the various areas of difficulty identified from the interviews and upon which the teaching programme was built are concerned, the programme appeared to have had some varied effects. The main gains for all groups appear to have been in the area of

Table 8.13

Percentage of Children Maintaining or Improving Their Level of Understanding (CSMS Algebra) Between Pre- and Delayed Posttest in Class Teaching (Other Teachers) Study

Number of CSMS Algebra Levels[a] moved	Year Group 1st (Age 12) n=19	2nd (Age 13) n=62	3rd (Age 14) n=16	4th (Age 15) n=25
0	47	60	50	28
1	53	31	50	64
2	-	8	-	8
Regressed	-	1	-	-

a. Note that the use of an abridged version of the CSMS Algebra test meant that only levels 0-3 could be assigned (see Appendix 1B for a description of the levels).

foregoing conjoining in algebraic addition, or in accepting the notion of an unclosed answer, an aspect which was also reflected in the gains generally observed in the formalization and symbolisation of method. The 'use of brackets' aspect was generally not well handled by this programme, but in view of the somewhat complex nature of this problem and the impossibility of dealing with all its aspects within the confines of the rather brief teaching module, this was perhaps not surprising. Of greater interest were the results obtained for the items concerned with the level of letter interpretation. That the programme could be effective on this issue was suggested by the gains on the relevant items observed for the two top stream second year groups, and, though to a lesser degree, for two of the fourth year classes. That the programme was generally not effective in this regard for the remaining groups is suggested to be worthy of further investigation.

Also of interest was the observation of high levels of specific errors among children not previously exposed to algebra, and the occasionally observed gains in performance over time in the absence of further teaching or practice in the topic. The possible implications of these findings, and the relevance of other, more particular, details for children's learning in elementary algebra, are discussed in the next section (Chapters 9 and 10).

PART 3

DISCUSSION AND IMPLICATIONS

CHAPTER 9: SUMMARY AND DISCUSSION

The clinical interviews and teaching experiments described in this thesis have led to the delineation of several areas of difficulty experienced by children in elementary algebra or 'generalised arithmetic'. In general, the teaching designed to help children overcome these difficulties has been shown to be successful, though with varying degrees of effectiveness. This chapter summarizes the research procedure and its main findings and discusses these in the light of other empirical and theoretical presentations, suggesting possible reasons for the variations in success of the teaching programme. In the next chapter, consideration is given to the implications of these findings for the teaching of algebra and for further research in this area.

Summary of Research Procedure

The research described in this thesis comprised three main phases, namely:

(1) an investigation of the causes of some particular errors in generalised arithmetic which the earlier CSMS (Mathematics) project had shown to be widely prevalent among second year (age 13) to fourth year (age 15) secondary school children, this investigation proceeding by means of individual interviews with children identified as making the errors in question;

(2) a series of small-group teaching experiments in which children's interaction with a teaching programme based on the information derived from the interviews was monitored; and

(3) the development and trial of an instructional programme derived from the findings of the small-group teaching experiments, designed for use with whole classes, and which aimed at helping children restructure their thinking so as to avoid making the errors under study.

A total of 72 interviews involving 55 children aged from 12 to 16 years from five schools was held in the first phase ('1' above), while the second phase ('2' above) was conducted in two schools with two groups of 14 year old children (each comprising five children), and one group of six 13 year olds. In the third phase ('3' above), four groups of children, comprising fourteen 12 year olds, sixteen 13 year olds, eight 14 year olds and twelve 15 year olds respectively (a total of 50 children) provided the data for the trials of the modified teaching programme derived from phase two and taught by the researcher, and the effectiveness of this programme with whole classes was assessed using seven classes (162 children) at first year (age 12) to fourth year (age 15) level in five schools, these classes being taught by their usual class teacher. (All the numbers given for phase three refer to the number of children in any group for whom complete sets of data were available.)

Summary of Findings

The areas of difficulty substantiated by the present research may be detailed as follows. These observations are, of course, limited to difficulties which appear to be experienced by children making the particular errors studied. Whether the same aspects do present problems to other children, but are by some means more successfully overcome in their case, has not been ascertained in the present study.

1. The interpretation of letters:

   (a) Children have difficulty in grasping the notion of letters as generalised numbers. Evidence from children who have not previously encountered algebra suggests that children may have a 'natural' tendency to interpret letters as standing for specific numbers, with different letters necessarily representing different values.

   (b) Whilst children may readily accept that letters represent numbers (albeit specific ones), they may nevertheless in some instances handle them as though they were entitites rather than quantities. This is particularly apparent in more abstract examples such as the simplification of 2a+5b+a, where children do not interpret the letters at all, but merely 'symbol push', inventing such rules as 'add all the numbers and then write down the letters'.

   (c) Some children are confused over the distinction between letters as representing the value(s) or number(s) relating to a measure or object, and letters as representing the measure or object itself. For example, children may interpret '6a' as '6 a's' or '6 apples' rather than as '6 times the number represented by a'. In line with this, children often interpret the particular letter used as initialling the object represented (e.g. 'a is for apples', 'y is for yachts').

2. The formalization of method:
   Children often do not understand the process of writing

down a formal statement of mathematical method in arithmetic, and consequently have no structure from which to generalise in the algebraic case. This difficulty appears to have several components:

(a) Many children do not make explicit the precise procedures by which they solve arithmetic problems. Consequently they are not aware of 'the method' as distinct from the problem to which it applies, and so cannot record it.

(b) The procedures which children use in solving arithmetic problems are often informal methods which are context-dependent and do not readily generalise to non-integer or algebraic cases.

(c) Children are often not precise in how they write (or read) mathematical expressions, since they tend to interpret them in terms of intended meaning (i.e. with reference to a given context) rather than literal or 'disembedded' meaning (cf. Donaldson, 1978).

(d) Children consider mathematics to be an empirical subject which requires the production of numerical answers. Even where children can formalize the required procedure and symbolise it correctly, they may not appreciate that this is an appropriate thing to do, i.e. they do not consider that a 'method' statement can also be an 'answer'.

3. The understanding of notation and convention:

(a) Largely as a consequence of the desire to obtain an 'answer' (single-term) to algebraic problems, children may attempt to perform an algebraic addition, producing the conjoined term as outcome (e.g. 'a+b' is replaced by 'ab'). That this association of the conjoined term and the algebraic sum is not the result of confusion with the algebraic product is shown by the fact that the same error is made by children who have not yet been taught algebra.

(b) Children ignore the use of brackets, mainly because they consider them unnecessary. This belief is largely founded upon the view that:

(i) the context of the problem determines the order of operations;

(ii) in the absence of a specific context, operations are performed from left to right;

(iii) the same value will in any case be obtained regardless of the order of calculation.

(c) Children show signs of other notational confusions. For example, 4y may be interpreted as '4 y's' (not the same as 4 times y) or as 'forty-y', or as 4+y (note 3a above). Hence for a given replacement value for y, such as y = 3, '4y' may be interpreted as 12, 7, or 43.

The results of the teaching trials indicated that a teaching programme based specifically upon the difficulties indicated above and designed explicitly to address the misconceptions upon which they are founded, can be successful (at least in part) in restructuring children's cognition with regard to the points at issue. Observations from the teaching and test results further indicated that:

1. Gains in performance may be observed in the period following the teaching programme (as in the period between immediate and delayed posttests), and these gains may be independent of further teaching or practice.
2. There may be maturation-linked factors contributing towards the child's likelihood of assimilating a given concept in the course of the teaching.
3. Many of the errors in elementary algebra investigated in the present research are also made by naive (in the sense of lacking previous exposure to algebra) pupils. The proportion of naive pupils making these errors was in many instances comparable to that observed for older 'more experienced' pupils.
4. Children may respond correctly to questions requiring the use of certain notation or convention (such as the use of brackets or the writing of an algebraic sum), and yet be unable to discriminate between correct and incorrect representations, i.e. be unable to select the correct 'equivalents' from a list which includes both correct and incorrect alternatives.

Discussion

General Observations

From the summary given in the preceding section, it can be seen that many of the difficulties which children appear to have in algebra, and which result in the kinds of error studied here, are not difficulties in algebra as such, but are rather difficulties in arithmetic. Many children (i) do not explicitly consider method in arithmetic, (ii) may in fact use procedures which are not formalized, (iii) tend to interpret expressions in terms of context to that the need for precision and rigor in the form of mathematical statements is not appreciated, and (iv) even in the case of those mathematical procedures of which they are explicitly aware are not always proficient in their symbolisation. Since what is usually emphasized in school arithmetic is the final numerical answer, many of these shortcomings go largely unnoticed, at least as far as problems involving small whole numbers are concerned, where a final answer can be obtained through more intuitive procedures. In algebra, however, as in more complex arithmetic cases involving large or non-integral values, the correct approach to problems requires precisely those processes and conceptions which the child may have successfully managed to avoid in dealing with simple arithmetic problems. In meeting algebra, therefore, these children are confronted with their own lack of understanding of arithmetic processes. In the same way, children's notational confusions in algebra, such as those caused by the omission of brackets, reflect the uncertainty of their understanding of number, as for example in their views on the order of operations by which the value of a numerical expression is regarded as constant regardless of the order in which it is written or even computed. By this view, algebra merely makes explicit the misconceptions and confusions which children already experience in arithmetic, but which may go largely unnoticed in that domain. At the same time algebra is not arithmetic, and important distinctions between the two may also account in part for the difficulties which children experience in this topic.

The Interpretation of Letters

The notion that children have difficulty concerning the meaning of letters is certainly not new and has been amply demonstrated by many researchers (see Chapter 4). Some of the misconceptions that children have may be due to inadequacies in the teaching-learning situation (for example, the notion that changing the letter implies changing the number represented, that letters which are later in the alphabet represent higher values (Wagner, 1981b), and/or the idea that the particular letter used is the first letter of the name of the quantity or object represented). Wagner (1981a) has discussed as many as nine different usages of letters in algebra, including letters as place-holders, (specific) unknowns, generalised number, indeterminates, variables, paramenters and constants, pointing out that these different usages may present quite different conceptual problems for the pupil, and that these should be taken into account in the teaching process. However, some of the difficulty which children have appears to be related more to a 'cognitive readiness' factor. This is particularly so with regard to the apprehension of letters as representing generalised number rather than specific unknowns. Part of this problem may, of course, be exacerbated by the teaching process, by which letters are typically introduced as representing specific unknown values, thus reinforcing a predilection which children appear to have 'naturally' (as evidenced by the pretest results from the first year teaching trials in the present research). However, the view that there is a deeper basis to this conception appears to be supported by the data which indicated a strong resistance, on the part of children in the present study, to the assimilation of the idea of letter as generalised number even within the context of a teaching programme specifically designed to address this aspect of algebra.

A possible basis for children's difficulty in understanding the generalised nature of algebraic values, and which is expressed in terms of the child's level of cognitive maturation, has already been outlined in Chapter 4. The findings from the present study do not, of course, permit any evaluation of the validity of that explanation. However, the results do suggest a possible association between acceptance of the notion of generalised number and the attainment of particular levels of reasoning. Certainly the observation that children typically experience considerable difficulty in coming to terms with the generalised nature of algebraic representation, and that there appears to be maturation-linked factor in children's ability to assimilate this notion (as indicated by the teaching trials in the present study, in which noticable gains in this direction were made only by the second year top ability groups and the fourth year middle ability groups - the two third year classes involved being of lower mathematical ability), is at least not inconsistent with the suggestion that the attainment of this level of conceptualisation is related to the development of 'higher-order' cognitive structures, perhaps in the manner derivable, as indicated in Chapter 4, from the Piagetian theory. However, to establish even this point more firmly requires a more systematic exploration of the notion of generalised number in terms of both a wider age and ability range of children and a more varied set of items relating to this issue (only two such items were used in the present study; the problem of designing items of this kind is not trivial and has been remarked upon elsewhere, e.g. Küchemann, 1980).

The Formalization of Method

Informal methods in algebra. The observation that part of the difficulty which children have in algebra stems from their use of informal methods and their inability to make the nature of the procedures used explicit, has been indicated by several researchers investigating children's solution of equations (e.g. Bell & O'Brien, 1981; Herscovics & Kieran, 1980; O'Brien, 1980; Petitto, 1979; Sleeman, 1982). Petitto's (1979) distinction between 'intuitive' and 'formal' approaches in equation-solving has already been discussed in Chapter 4. Petitto also drew attention to the fact that children often did not formalize exactly what they had done in the 'easy' case, so that they were unable to apply the same method to the harder example even if this had been appropriate:

"Interviewer: What were you trying to do?
BC: Tried to do it the same as as I did that one (the easier one).
Interviewer: O.K.
BC: But I forgot how I did that one ...."
(Petitto, 1979, p.77)

Petitto also noted that some children switched from formal to informal strategies or vice versa, but indicated that the two approaches were not necessarily seen as being related:

"Interviewer: Can you take a guess what it might be?
HL: Well, if I did it without trying to do it mathematically I problably could."
(ibid, p.80),

a remark which seems to be closely akin to that made by a child interviewed in the CSMS research when trying to explain an incorrect answer she had produced: "I was doing it mathematically, not logically." (Hart, 1978, p.6). Bell and O'Brien (1981) also noted that children's equation-solving strategies often change with the demands of the question, and Herscovics and Kieran (1980) pointed out that children's ability to solve simple equations was often dependent upon their ability to 'review the arithmetic facts stored in their memory', so that their obtaining of an answer amounted to mental arithmetic rather than any formal algebraic process (p.573). Sleeman (1982) observed that children had difficulty in solving more complex equations because they were endeavouring to use the same method of solution by inspection (or 'guess and test') which they used, often successfully, in solving simple equations. The difficulty in solving the more complex problems therefore seemed to be related to the fact that children did not adequately formalize the procedures used in the simpler case, and were in fact using informal procedures in the latter instance which were difficult to represent in a manner which could be generalised to harder examples.

Informal methods in primary mathematics. From the findings discussed in the previous section and from those of the present research, it seems possible that many children at secondary school level may be solving mathematical problems in a more intuitive manner without appreciating the appropriate general relationships. Where primary school children are concerned, the notion that children may develop and adhere to methods of their own in mathematics is certainly not new. With reference to young children's computational procedures, for example, Brownell (1935) had commented that "in spite of long-continued drill, children tend to maintain the use of whatever procedures they have found to satisfy their number needs" (p.186), implying that some of their procedures were different from those which were the subject of drill. More recently, Plunkett (1979) contrasted the marked lack in variety of school-taught algorithms with the wide diversity of computational procedures which children actually use, making reference to an investigation by Jones (1975) which obtained 16 different ways of calculating 83 - 26 from 80 eleven year old children. The use by young children from pre-school to third-grade level of 'invented' addition procedures and algorithms had also been observed (e.g. Groen & Resnick, 1977; Groen & Parkman, 1972; Houlihan & Ginsburg, 1981; Russell, 1977; Suppes & Groen, 1967), and the dependence of these on counting procedures noted. Like Groen and Resnick (1977), Houlihan and Ginsburg (1981) suggested that children's informal procedures are based partly on what is taught in school, and partly on their own invention: "children assimilate what is taught into what they already know and the result is the 'invented strategy'" (p.104). Other studies of young children's solution of addition and subtraction word problems have similarly revealed a range of pre- or un-taught informal strategies (e.g. Carpenter, Hiebert & Moser, 1983; Carpenter & Moser, 1982; Ibarra & Lindvall, 1982; Moser, 1980). Ginsburg (1975, 1977a,b) has drawn particular attention to a view of the young child as an 'intuitive mathematician', who develops an informal knowledge of mathematics which is then used as a basis from which arithmetic problem-solving methods are derived, these often seeming quite different to those taught at school. Ginsburg further suggested (1975) that teaching could not proceed effectively without attention to the child's informal knowledge, a point particularly stressed also by Case (1975, 1978). Case drew attention to the fact that "young children tend to develop reasonable but oversimplified strategies" (1978, p.442) for solving tasks, and suggested that part of their difficulty in learning was due to their "inability to overcome the natural way of responding to a situation" (1975, p.68). Consequently, Case considered that in order to be effective, teaching must take into account both the 'incorrect' and correct response, and that instruction must be designed so that the limitations of the natural strategies would be apparent:

> "Until one understands what children do spontaneously, one will not be able to demonstrate the limits of this approach to them. Furthermore, until one does demonstrate the limits of whatever approach students use spontaneously, they will not thoroughly understand the necessity for using the approach to be taught."
> (Case, 1978, p.433).

This difference between what the teacher teaches and what the child often does was also one of the main issues discussed by Ginsburg's colleague Erlwanger, as the result of a series of case studies into children's conceptions of mathematics (Erlwanger, 1973, 1975). Erlwanger's interviews with children clearly revealed that elementary school children employ a variety of methods, many of which were not taught, and which the child should have abandoned after instruction in 'more mature ways' (1975, p.163). He furthermore suggested that the child often did not perceive any relationship between his own intuitive ideas and those of school mathematics, and often developed a resistance to accommodating the taught procedures and perspectives, a possibility also suggested by Hart (1980a; see also Booth & Hart, 1983; Hart & Booth, 1981):

> "it seems likely that children do not assimilate a new method (or algorithm) if they find that they can manage (even if only for a limited period) with their own .... methods."
> (Hart, 1980a, p.9).

<u>Alternative conceptions in secondary science.</u> Although well documented at primary school level, the widespread use of informal methods among secondary school children remains to be established. However, it can be seen that the possibility of such an occurrence

could have profound consequences for children's understanding and performance in algebra as well as in mathematics generally. In secondary school science, the existence of 'primitive' conceptualisations or 'alternative frameworks' (Driver, 1981) held by school-children has been well documented (e.g. Andersson,1980; Champagne, Klopfer & Anderson, 1980; Erickson, 1979; Kargbo, Hobbs & Erickson, 1980; Novick & Nussbaum, 1978; Nussbaum & Novak, 1976), and evidence presented for the apparent persistence of such conceptualisations despite instruction (Champagne, Klopfer & Anderson, 1980; Gunstone & White, 1981; Rowell & Dawson, 1977). Unlike the 'alternative frameworks' described in science, however, the informal procedures apparent in mathematics appear to be essentially correct, and indeed often form the basis from which the more formal procedures are constructed. The problem associated with them is not one of incorrectness, but rather of limited applicability. Nevertheless, in mathematics as in science the consequences of the pre-existence of these 'alternative frameworks' can be profound: faced with an instructional sequence which is not in keeping with the child's own conceptual framework, the child has to either reconstruct his own conceptualisation, or keep two separate systems (Driver & Easley, 1978). Whilst perhaps familiar to science educators, however, this idea of the possible co-existence of two 'cultures' within the classroom and the likelihood of the child's own system interfering with assimilation of the taught system, is by and large still relatively new to mathematics teachers. By means of its investigation into the conceptions and strategies underlying children's errors in mathematics, the SESM project (of which the present research is a part) has provided strong support for earlier research providing evidence for such an explanation of many children's lack of progress in formal mathematics. It is the perseverance of firmly held 'intuitive' notions of mathematical concepts and procedures on the part of these children that hinders their assimilation of the more formalised concepts and procedures which are the substance of school mathematics.

'Intuition' and formal procedures. The existence of intuitive notions in mathematics are, of course, essential to the child's understanding of mathematical concepts and procedures as Skemp (1968, 1971) among others has pointed out. However, that 'intuition' can have a negative as well as positive effect on mathematical thinking has been remarked upon also by Fischbein (Fischbein, 1979; Fischbein, Tirosh & Hess, 1979). Discussing the self-evidence of various mathematical statements, Fischbein (1979) makes the point that this fact can both help children to understand the statements, and at the same time provide an obstacle to the apperception of the logical structure of mathematics:

> "Why is it necessary to prove that in a rectangle the diagonals are equal if it is absolutely evident that it is so? The intuitive obviousness of such a statement simply blocks the acceptance of the utility of the corresponding logical proof."
>
> (Fishbein, 1979, p.152)

Similarly, the ease with which young children can solve addition and subtraction problems by using counting techniques may deter them from struggling with the complexities of the addition and subtraction algorithms, or indeed from seeing the use of the latter. On reviewing college students' attempts at symbolising algebraically the relationship presented in a simple word problem, Clement (1982) also referred to the existence of intuitive conceptualisations or procedures in mathematics, and to the possible 'conflict' between these and the more formal taught procedures, pointing out that even if the latter are taught successfully (in terms of the students' observed adoption of those methods), the more primitive procedure may still re-assert itself at a later date:

> "students who are successfully taught a standard method for a mathematical skill may still possess intuitive, nonstandard methods that can compete for control..... This implies that teaching a student a standard method is no guarantee that the student's own intuitive method will not 'take over' in a later problem solving situation."
> (Clement, 1982, p.28)

This possibility of later reversion to an earlier procedure has also been indicated by Davis (e.g. 1980). Drawing particularly upon the work of researchers such as Clement as well as upon their own findings, Davis and his colleagues have suggested that students can deal with mathematics on at least two different levels, a surface level and a deeper, more intuitive level. While initial teaching may produce a correct response at the surface level, there is no guarantee that this has been accompanied by a corresponding correct 'deeper' level processing. Where this has not occurred, the surface response will disappear in time and the earlier mode of operation, which derives from the deeper structures, will reappear.

A 'cognitive' explanation - Collis' interpretation. If the use of informal methods by children is as widespread as the research indicates, then it becomes of interest to know why this should be the case. Children are certainly taught the formal procedures, so that the continued use of informal approaches must proceed despite such teaching. In attempting to explain this phenomenon, reference may perhaps again be made to Collis' suggestion that differences in mathematical approach may be related to the Piagetian distinction between concrete and formal operational thinking. Collis reminded us that the concrete operational child's thinking is restricted to concrete-empirical experience so that the child tends to operate in terms of the particular situation presented, and does not obtain a content-free grasp of the structure of things. Consequently the concrete operational thinker has difficulty in working in an abstract mathematical system. From many points of view, all (formal) mathematical activity, and certainly much of what constitutes school mathematics, requires children to work within 'an abstract mathematical system'. While the teacher is concerned with the abstraction of general mathematical principles and relationships which can be operationalised in the development of formal procedures (and these generalised to the solution of a whole class of problems), however, the child does not look beyond the particular solution of immediate, concrete, problems. The distinction between the comprehension and use of the formal taught procedures and the use of specific informal methods dependent upon the particular context of a problem, may therefore be associated with the distinction between formal and concrete operational thinking in the Piagetian sense. The importance of context in influencing both how the child solves the problem, and how the child interprets mathematical expressions, has been demonstrated in the present research. Nevertheless, the finding that children can gain from instruction aimed at assisting them to move towards operating in the more formal system suggests that the conceptual demands of this task are not beyond children in the age range studied. This observation seems to be at variance with the notion that this ability awaits the development of the same (formal operational?) level of thinking as does the notion of generalised number. This point is touched upon later in the chapter.

Notion and Convention

Conjoining in arithemtic and algebra. Although the suggestion has been made that much of children's difficulty in algebra reflects difficulties of formalization in arithmetic, and that algebra in many ways involves the making explicit that which is implicit in arithmetic (albeit perhaps not even implicit for many children), algebra is not simply a straightforward generalisation of arithmetic (cf. Matz, 1980). The shift to algebra represents a shift from concern with specific numerical context and as such is new to the child, and has consequences which cover issues of symbolism as well as the whole question of the nature and purpose of algebraic as opposed to arithmetic activity. While, for example, algebra may be regarded (at least in part) as a generalisation of arithmetic statements, the representational rules used differ in important respects from one domain to the other. Thus in arithmetic, conjoining denotes implicit addition in both the place value sense (e.g. 43 = 4 tens + 3 units) and the representation of mixed fractions (e.g. $2\frac{1}{2} = 2+\frac{1}{2}$). In algebra, however, conjoining denotes not addition, but multiplication (e.g. 3a = 3xa). Discussing this distinction, Matz (1980) has suggested that some of the errors which children make in algebra are due to the false generalisation of rules which are appropriate in the arithmetic but not algebraic case. The observation that children with no previous exposure to algebra often use conjoining to symbolise the algebraic sum may provide support for this suggestion.

Letters in arithmetic and algebra. The use of letters appears in both arithmetic and algebra, but in quite different ways. The letters m and p, for instance, may be used in arithmetic to refer to 'meters' and 'pence' rather than representing the number of meters or the number of pence, as is the case in algebra. Some of the difficulties which children have in interpreting algebraic terms may well relate to this change in usage between arithmetic and algebra. '3m' for example, is read in arithmetic linguistically as '3 meters', and statements such as '3m = 300cm' are statements representing a static relationship and are also translated linguistically as '3 metres are equivalent to 300 centimetres'. In algebra, however, statements of this kind are not read linguistically but rather are 'dynamic' statements expressing 'active operations' (see Clement, 1982). This difference in usage may account for some of the difficulty which children appear to have in dealing with algebraic statements. For example, a response often given to the question 'write an algebraic statement to show the relationship between students and professors given that there are six times as many students as professors' is '6S = P' (see, for example, Clement, 1982; Rosnick, 1981). The giving of such a statement for this problem is consistent with an arithmetic interpretation, in the manner described above, of 'six students are equivalent to one professor' or 'for every six students there is one professor', but it is not correct when viewed algebraically. In the latter case the expression means '6 times the number of students is equal to the number of professors', and so is not a correct description of the given relationship. Again, some of the difficulties which children have in the algebraic case may reflect not so much a lack of arithmetic knowledge but rather the attempt to apply arithmetic notions which are apprehended to the algebraic situation in which they are not appropriate.

The acceptance of unclosed answers. One of the most obvious distinctions between arithmetic and algebra, of course, lies in the concern with numerical answers. In the case of arithmetic, a model of addition (for example) can be presented which is 'some sort of coffee grinder ... you throw, say, "5" and "2" into the hopper ... and finally a "7" comes out ...' (Davis, Jockusch & McKnight, 1978, p.17). An algebraic expression, however, cannot be evaluated in this way; consequently by convention the result of the operation is labelled by the 'procedure call' (Matz, 1980, p.132). The result of adding 7 and x, for example, is represented as 7+x. This means that, in order to work with algebraic elements, pupils must accept both the idea of unclosed operations, and the notation for representing them, which in turn means "relaxing arithmetic expectations about well-formed answers, namely that an answer is a number" (Matz, 1980, p.132). This notion may not be an easy one to accept. Davis, Jockusch and McKnight (1978) have noted that children frequently have difficulty with the idea of adding 7 and x:

> "This is one of the hardest things for some seventh-graders to cope with; they commonly say: 'But how can I add 7 to x, when I don't know what x is?'"
> (op. cit., p.100).

As mentioned earlier, Collis had drawn particular attention to the idea that children vary with respect to their 'acceptance of lack of closure' (Collis, 1972; see also Lunzer, 1973), or their ability to 'hold unevaluated operations in suspension' (Kieran, 1981a, p.319), considering it to be an ability indicative of the formal operational stage. From this point of view, children's ability to accept the notion of unclosed operations may be more dependent upon the attainment of a particular level of cognitive maturation rather than upon instruction per se. Wollman, Eylon and Lawson (1979) have criticised this particular view, considering that many subjects' reluctance to accept lack of closure relates more to their perception of what is required, rather than to any incapability in this direction. The whole of school experience, Wollman and his colleagues point out, works against any tendency to suspend closure, and leads subjects to presume that they are supposed to give an unequivocal answer.

The meaning of symbols. Evidence from several studies of young children's handling of arithmetic operational statements shows that they are unwilling to accept open number sentences as 'answers', or even to leave them uncomputed (e.g. Behr, Erlwanger & Nichols, 1980; Ginsburg, 1977a). Ginsburg (ibid) related this occurrence to children's views of the symbols (e.g. '+' and '=') used in such statements, pointing out that children interpret such symbols in terms of actions to be performed, so that '+' means to actually perform the operation and '=' means to write down the answer. For Ginsburg, therefore, young children's reluctance to accept lack of closure is related to the way in which the symbols involved are viewed by the child. Behr, Erlwanger and Nichols similarly showed that children regard such symbols as + and = as a 'do something signal' (Behr, Erlwanger & Nichols, 1980, p.15), and noted that this was as true of 6th grade children (age 12) as of 1st graders (age 6) (see also De Corte & Verschaffel, 1981). Such an interpretation does not seem to be restricted to primary school children: thus Kieran (Kieran 1980, 1981a, b; Herscovics & Kieran, 1980) showed within the context of equations and arithmetic identities that secondary school children aged 12 to 14 years typically regard the equals sign as a unidirectional symbol which precedes a numerical answer, and Wagner's (1977) 17 year old students were likewise observed to regard the equals sign as a 'do something signal'. The idea that the equals sign be viewed as signalling a relationship rather than an operation, while perhaps essential to algebraic understanding, may be one which does not come easily to the child. Consequently, a move from arithmetic to algebra necessitates, besides a change in representational interpretation and a modification of the concept of operations such as addition, a possible change in view of the equals sign, and of the whole purpose of mathematical activity, which is now no longer the seeking of a specific numerical answer.

Closure and cognition. The present research noted that the idea of an unclosed, non-numerical answer was initially not accepted by children in the age range investigated here, namely 12 to 15 years. However, the apparent effectiveness of the teaching programme in restructuring children's thinking in this regard would suggest that the notion was not beyond the conceptual grasp of these children. It also suggests that the acceptance of lack of closure, and the view of letters as generalised rather than particular number, may relate to different levels of conceptual difficulty, rather than be manifestations of a single cognitive structure as suggested by the Collis-Piaget formulation.

The use of brackets. The issue relating to the use of brackets has been the subject of relatively little investigation. Kieran (1979a) drew attention to the fact that children typically do not use brackets, considering their use unnecessary since for children the written sequencing of operations defines the order of computation. The influence of context in varying this interpretation, and the expectation that the value of an expression remains unchanged even if the order of computation is varied, has been indicated in this study (see also Booth, 1982d, in Appendix 13).

<u>The problem of symbolic representation.</u> It is clear from the above discussion that the question of symbolic representation is in itself not a trivial one. In particular, the nature of the kind of understanding of symbolic representation that a child has may require consideration, as will the child's ability to hold dual interpretations of certain symbols and to shift readily from one interpretation to another. The finding in the present research that many children were able to consistently produce a correct symbolic representation (as in the case of the use of brackets or the production of an unclosed answer), yet did not discriminate between correct and incorrect representations when presented with a choice, suggests that the consolidation of symbolic meaning may itself progress via stages. The theoretical debate concerning the process by which the child comes to acquire knowledge of symbolic representation in mathematics or in language is an extensive one whose consideration is beyond the scope of this thesis. Certainly mathematics can be regarded in part as "a special kind of language developed from natural experience, but then formalized into an independent conceptual system with its own rules, symbols and meanings" (Beilin, unpublished, quoted in Ginsburg, 1975, p.137). The processes by which the child comes to learn this formal language cannot be taken for granted but do themselves require careful and detailed analysis. Vergnaud (1982) has drawn particular attention to the problem of the role played by symbolic representation in mathematics, and asked how teaching can address itself to this issue. He suggests two main procedures in this regard, firstly that instructors must try to "make operational the problem of symbolic representation by finding tasks in which a

symbolic representation helps pupils to solve the task" (p.349), and secondly that "it is sometimes better to differentiate between situations that are not the same for the child although they do come to the same numerical equation" (p.350). The first ask is not so easy, as Vergnaud points out, since children often solve the problem first and then write its representation (if at all) afterwards. What is needed is a choice of task and symbol systems so that the former cannot readily be solved without the help of the latter. The value and purpose of symbolic representation is thus made apparent, and its meaning acquired by its constructive use. As regards the second point, it is suggested that the attainment of the most abstract formal meaning of symbols may be best approached by means of an intermediate step in which children use differentiated symbols to represent the different interpretations normally associated with the same mathematical sign (for example, using different symbols to signify the 'write down the answer' and 'is equivalent to' meanings of the equals sign). Other procedures may also be useful in helping children acquire symbolic meanings; this point will be referred to again in the next chapter.

Theories of Cognition and Children's Understanding of Algebra

In Chapter 3, it was maintained that Piaget's theory of cognitive development could usefully inform the present research from a theoretical as well as methodological point of view. It was also suggested that the 'framework of knowledge' which a child possessed relevant to a given area of study would also have implications for the child's level of functioning within that area. It will be useful to reassess those beliefs in the light of the findings obtained in the present research.

In order to be usefully applied to those findings, any theory of cognition must account for (i) children's difficulty in accepting the notion of letters as generalised number, (ii) their unwillingness to accept unclosed answers and (iii) their apparent inability to obtain a context-free grasp of the mathematical structure of problems, i.e. to dissociate the 'method' and the precise way in which it is symbolised from the features of particular problems.

A Piagetian view. The Piagetian formulation of concrete as opposed to formal operational thinking seems immediately to account for all these observations. As discussed both in this chapter and in Chapter 4, it may be inferred from Piaget's theory that concrete and formal operational thinkers would differ in precisely these aspects, namely their view of the meaning of letters in algebra, their requirement for closure of operations, and in their ability to look beyond the features of a particular situation in order to obtain a content-free grasp of the structure of things and so, for example, work meaningfully within an abstract mathematical system. The picture drawn in the present study of the difficulties experienced by children in elementary algebra is thus consistent with a 'Piagetian' analysis of concrete operational thinking as it applies to the study of mathematics.

An 'alternative frameworks' model. There may, however, be an alternative explanation for these findings. Prior to their introduction to algebra, children have been working within an arithmetic 'framework of knowledge' (see Matz, 1980) in which (i) 'closed' (numerical) answers are required, (ii) the problem of the mathematical structure of problems has been subordinated to the obtaining of a correct answer by means of whatever procedure makes sense, (iii) conjoining does imply addition (in both the place value sense and the representation of mixed fractions), (iv) symbols representing quantities always signify unique values, and (v) letters are commonly used to refer to an object or measure rather than its quantification (as in '6m = 6 metres'). In the absence of any alternative (more appropriate) framework of knowledge, it would not be surprising if, on meeting algebra, children adhered to the 'rules of the game' already established for the arithmetic case, with inevitable consequences as far as their algebraic performance is concerned. The same kind of explanation may also account for other examples of apparent errors in reasoning which have been associated, in the Piagetian view, with a specific 'cognitive stage'. For example, young children's demonstrated lack of number conservation may be due not so much to the lack of attainment of a particular cognitive 'stage' structure, as to the fact that the framework of reference within which they have been working does not dissociate numerosity from space occupied. Ginsburg (1975) has discussed this issue in particular, drawing attention to various studies showing that very young children can distinguish numerosity beyond their counting range. This achievement appears to be based upon the fact that the child deals not with number as such, but with space occupied, or density. This particular viewpoint promotes a strategy which often brings the child success in judging relative 'quantities', but which is unfortunately not appropriate in judging conservation situations where the space occupied/density configuration is deliberately distorted. The successful handling of the number conservation task, therefore, requires that the child's framework of reference be modified to admit the discrimination of number from space occupied, i.e. requires a change in viewpoint and the 'rules of the game' (cf. Donaldson, 1978). Consequently, it seems that the same findings that are so cogently accounted for by the Piagetian theory may also be explained by reference to a 'framework of knowledge' model, as in the case of the present research. Is it possible to discriminate usefully between the two?

Observations from the teaching experiment results. The results from the teaching phase of the present study may help throw some light on the possibility of distinguishing between the role of general cognition and particular knowledge frameworks. Observations from the teaching experiments supported the view that children have difficulty in understanding the notion of letters as generalised number. As a result of the teaching programme designed to address this issue, however, gains in understanding of this notion (as measured by improved performance on items selected as relating to this notion) were obtained (see for example, Figure 8.4). Of particular interest in this respect was the observed improvement in performance over the two months or more between the immediate and delayed posttests, and which occurred in the absence of further teaching or practice in this topic. These increases in performance with time seem consistent with Piaget and Inhelder's (1973) theory of a constructive memory, which postulates that memory can actually improve with time since it may permit the learner to construct certain connnections not immediately noticed in the first place. By this view:

> "the act of comprehending and encoding into memory is a Piagetian assimilation-type proces of construction ... similarly, retrieval is conceptualised as an equally active and assimilative process of reconstruction rather than as a passive unedited copying out of what is stored in memory."
> (Flavell, 1977, quoted in Herscovics, 1979, p.106)

Hence the observation of improvement in performance over time and in the absence of further practice or instruction may indicate the construction of memory or other appropriate cognitive structures which take time to become established. This suggestion may, of course, apply equally well to the construction of frameworks of knowledge, however.

Also of interest with respect to the observed changes in performance on items relating to the notion of letters as generalised number, was that the improvements noted were obtained only by the second year top stream classes and the older (fourth year) children who were described as 'good CSE students'. In the case of the younger (first year) children, and the second to fourth year children who were in lower mathematical ability groups, there was virtually no gain in performance in this domain (see, for example, Figures 8.13 and 9.1). Whilst the range of groups used in the present study did not allow this issue to be systematically investigated, the suggestion remains that there may be a maturation-linked 'readiness factor' associated with the acceptance of the notion of generalised number. Such a possibility is, of course, consistent with the 'Piagetian' view that this level of conceptualisation is linked to the attainment of formal operationality.

The same results, however, were not observed in the case of acceptance of an unclosed answer. In this area all classes made notable gains, regardless of age or level of mathematical ability (see, for example, Figure 9.2). The apparent effectiveness of the teaching programme in restructuring children's thinking in this regard would suggest that the notions involved were not beyond the conceptual grasp of these children. It also suggests that the acceptance of lack

Item: L + M + N = L + P + N True: Always/Never/Sometimes, when......

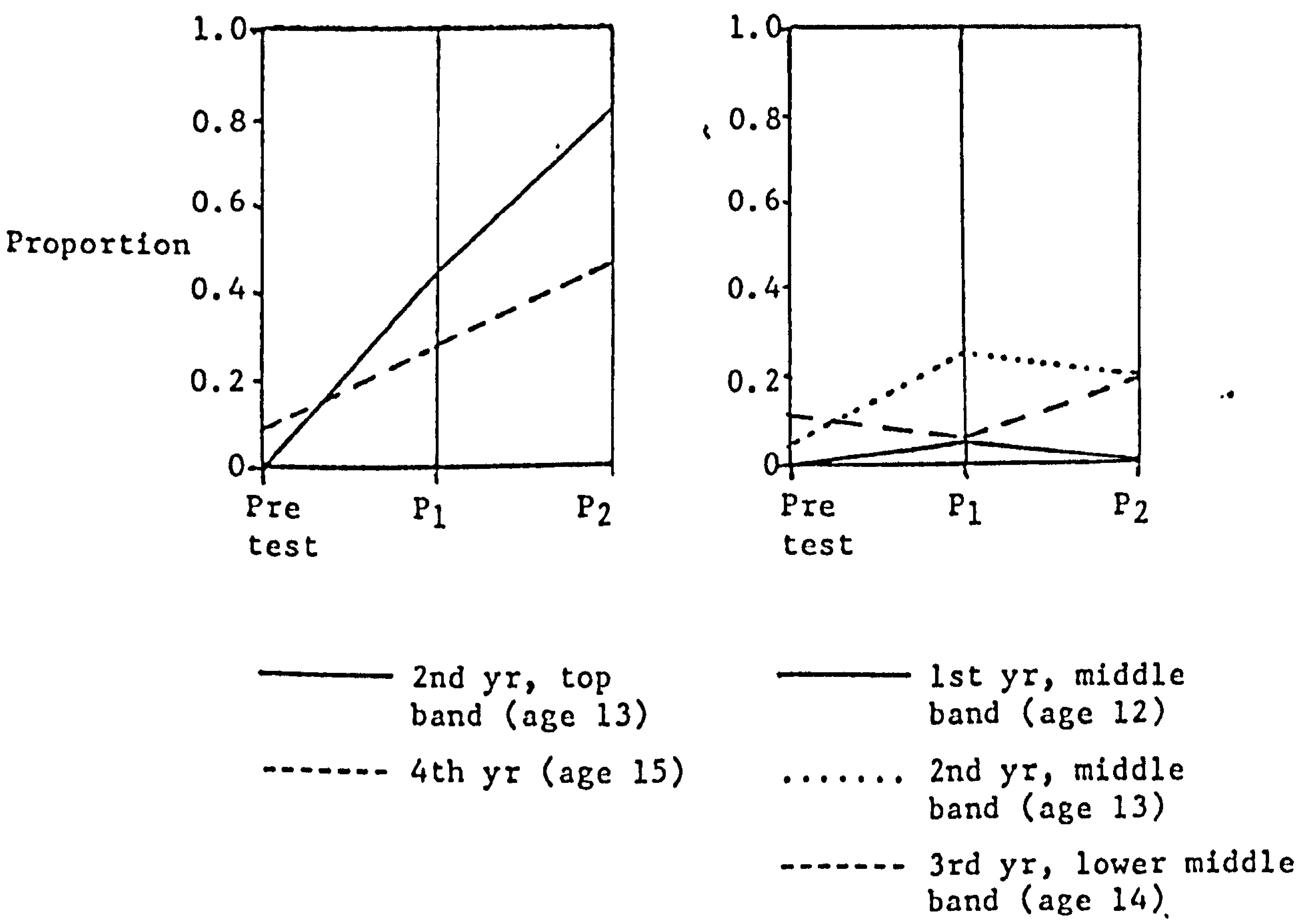
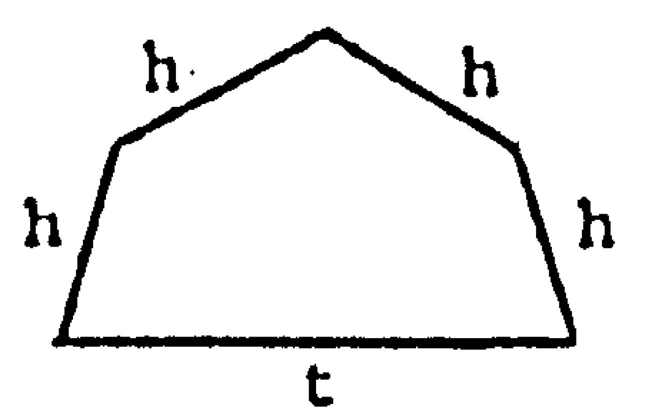


Figure 9.1 Proportion of children giving correct answers to item relating to 'letter as generalised number' on pretest, immediate posttest (P_1) and delayed posttest (P_2): Comparison between groups of different 'cognitive maturity'.

Item: Perimeter:



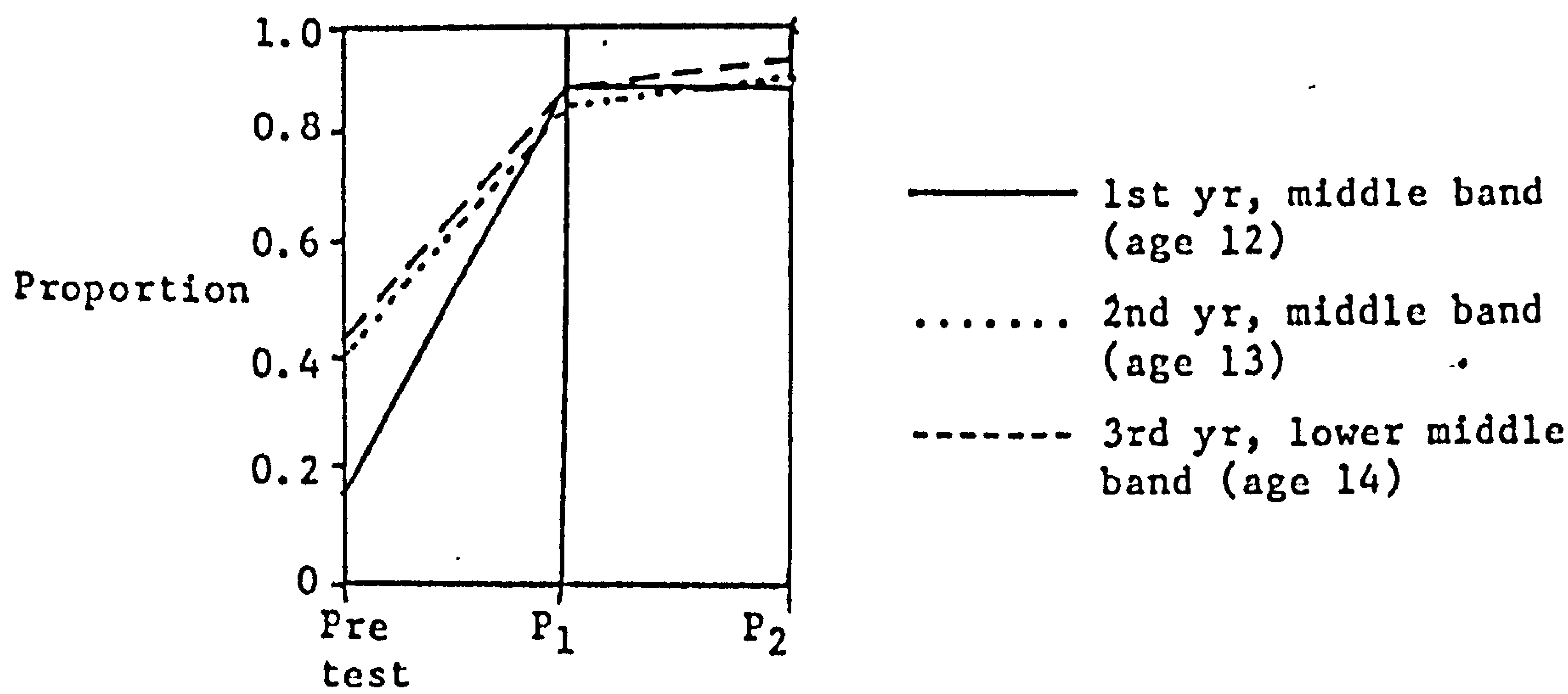


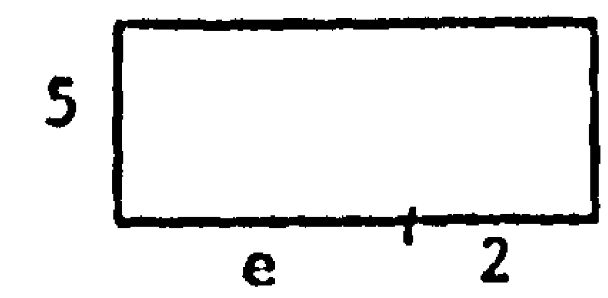
Figure 9.2 Proportion of children giving correct answers to item relating to 'conjoining in algebraic addition' on pretest, immediate posttest (P_1) and delayed posttest (P_2): Comparison between groups of different 'cognitive maturity'.

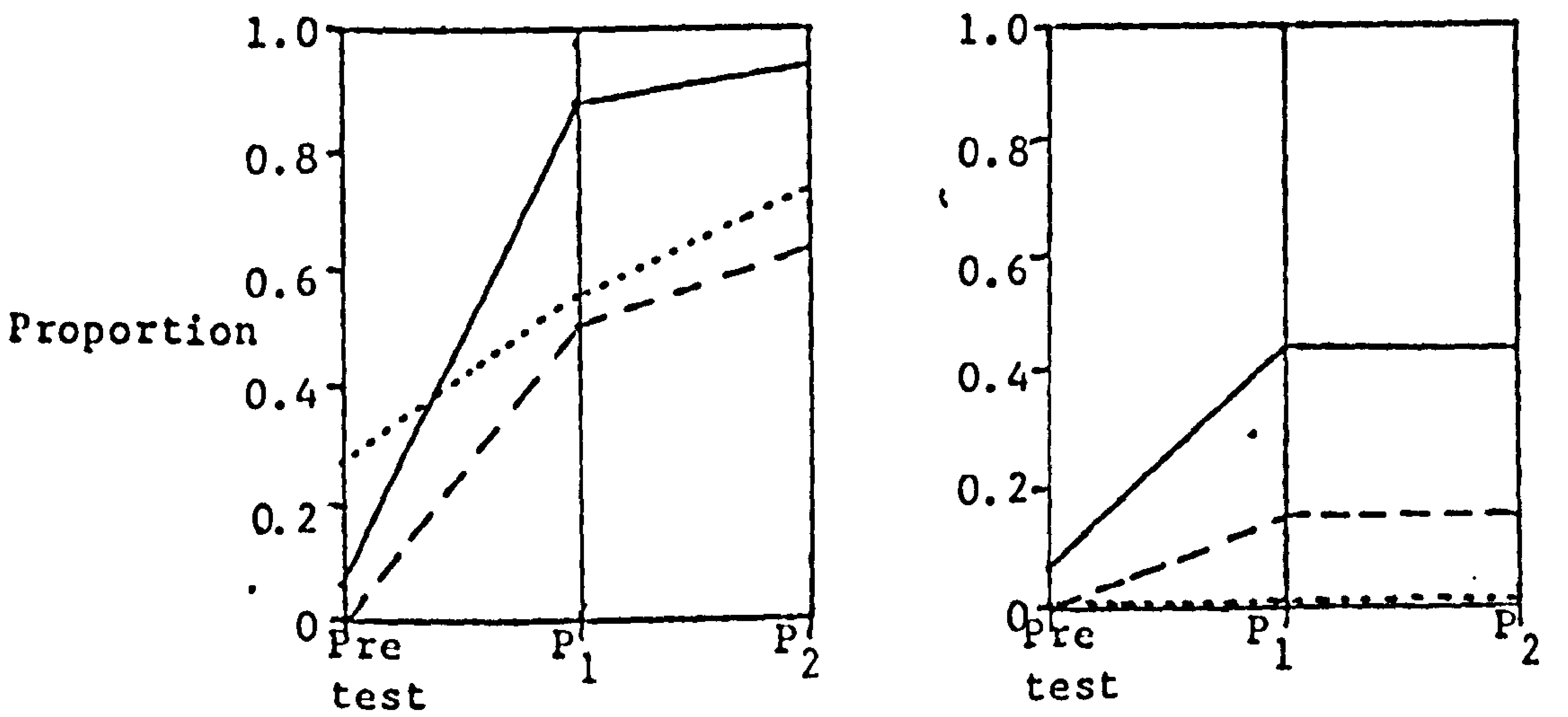
of closure, as defined in the present context, may not relate to the same level of conceptual difficulty as the notion of letters as generalised number. This would seem to imply either that acceptance of lack of closure is not a function of formal operational thinking (see, for example, Wollman, Eylon & Lawson, 1979 with respect to the comment that ALC is not an index of formal operational reasoning), or that the Piagetian 'unified stage' view that all indices of formal operational thinking appear simultaneously is again questioned (Brown & Desforges, 1977, 1979). Alternatively, of course, the suggestion made in the present research that the tasks under consideration are relevant to the concrete-formal operational distinction may be invalid, in which case the consistency referred to earlier between the difficulties identified in this study, and the 'Piagetian' (as interpreted here) analysis of concrete operational thinking is merely fortuitous.

The results with regard to the formalization of method are somewhat more equivocal. In some cases (i.e. some items), notable gains in performance were obtained and these observed for all classes regardless of age or ability level. In other instances, however, the picture is less clear, with an initial improvement in performance being followed by some later decrease in performance, at least in the case of the younger or 'less able' children, or indeed with there being little observed improvement at all (see Figure 9.3). However it may be the case that this latter observation reflects not so much a difficulty in formlizing method as a difficulty in correct symbolisation of that method. This suggestion receives some support from two observations. Firstly, the formalization of method items which appeared to show the smallest improvement in performance tended to be those which were notationally more complex, such as the item

Item: Perimeter: n sides, each of length 2

Area:





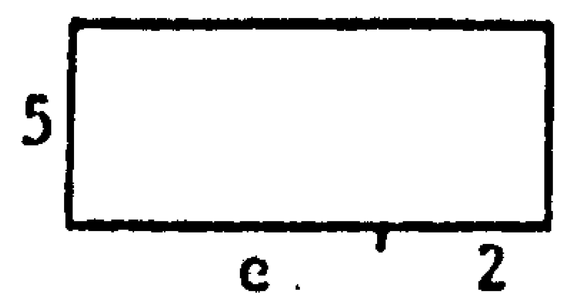
——— 2nd yr, top band (age 13)	——— 2nd yr, top band (age 13)
------- 3rd yr, lower middle band (age 14)	------- 2nd yr, middle band (age 13)
.... 4th yr (age 15)	.... 2nd yr, lower band (age 13)

Figure 9.3 Proportion of children giving correct response to items relating to 'formalization of method' on pretest, immediate posttest (P_1) and delayed posttest (P_2): Comparison between groups on two different kinds of item.

requiring the area of a rectangle measuring 5 by e+2 units, in which a correct answer often needs the use of brackets. Secondly, while little gain in performance in terms of obtaining the correct answer might have been observed, in many cases there was nevertheless a notable decrease in incidence of 'error' responses of the kind which signalled the avoidance of a 'method' statement, such as the giving of an alphabetic or numerical answer (see Figure 9.4). This suggests that children had moved towards the giving of an algebraic answer (and indeed this was shown to be the case by an inspection of their scripts), but were as yet not competent to symbolise that answer correctly. Consequently it may be that even in these instances the children had improved in their ability to formalize method, but still lacked the appropriate knowledge to permit a correct symbolisation of that formalization. In terms of the task given (namely the completion of written test papers) it was, of course, not possible to dissociate these two issues of formalization and symbolisation in order to assess their relative contributions to the results obtained.

It is, therefore, not possible to draw any clear picture from these results with regard to the notion that ability to formalize method is related to the attainment of formal operational thinking. The observation of some maturation-linked differences in performance in this regard may support this view, but the finding that marked gains in performance were in other instances obtained for all groups of children mitigates against it. The involvement of symbolism-linked as well as formalization-linked aspects in successful performance of the tasks involved does not help clarify the issue; consequently, more work would be needed in order to resolve this question.

Item: Area:



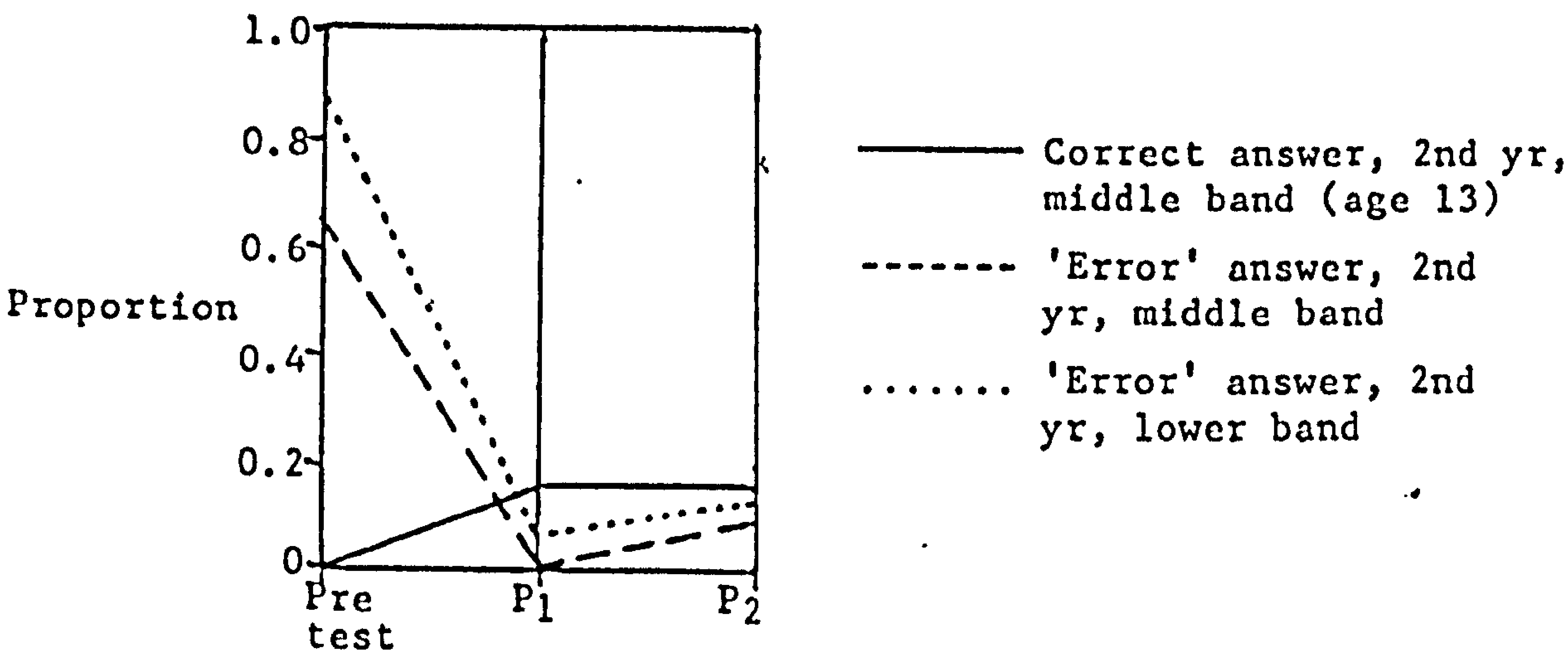


Figure 9.4 Proportion of children giving correct and 'error' responses to item relating to formalization of method on pretest, immediate posttest (P_1) and delayed posttest (P_2): Illustration of item showing little gain in proportion of correct responses but large decrease in proportion of related error.

The implication of a 'framework of knowledge'. However, in general the results seem to indicate that the children were able to benefit from the relatively brief intervention occasioned by the teaching programme in order to improve their performance with regard to acceptance of lack of closure and, at least in some instances, the formalization of method. The relative ease of assimilation of the notions of acceptance of lack of closure and of isolation and representation of method apparently demonstrated in the present teaching experiments does suggest that the cognitive structures necessary for such assimilation were already available to the children. It therefore becomes of interest to know why these children had not in fact already acquired these conceptualisations, but were instead making errors symptomatic of their absence. It is suggested that this occurrence derives not from any cognitive inadequacy on the children's part, but rather from the inappropriateness of the framework of reference within which they are working. As indicated earlier in this discussion, the 'arithmetic' framework of reference appropriate to children's prior mathematical experience is consistent with the requirement for closed answers, does employ conjoining to denote addition, and rarely focusses attention on the precise recording of statements of formal mathematical method. Enabling the child to contruct a more appropriate framework of reference with regard to these issues in an algebraic context may therefore be expected to remove much of the difficulty associated with this particular source. Certainly it seems that if the change in 'rules of the game' needed for correct algebraic functioning is acknowledged by the child (as, for example, in the teaching module developed in the present study), then much of the 'conceptual difficulty' of such notions as acceptance of lack of closure and contemplation of formal method or structure may vanish. By promoting such a change in framework of reference, it is suggested that the teaching programme utilised in this study was successful in helping the children to improve performance on items relevant to those potential areas of difficulty.

The implication of 'cognitive maturation'. The apparent resistance on the part of children observed in this study to the notion of letters as generalised number seems to indicate the involvement of maturation-linked cognitive factors in addition to the establishment of more appropriate framework of reference. In addition, the observed similarity in the nature of the 'child-methods' used by different children in response to a variety of problems seems to indicate some communality in logic, or at least of approach, which requires recourse to general cognitive abilities for explanation. Finally, of course, whilst the child's particular framework of knowledge may have importance for the level at which the child will function in that domain, it is still necessary to explain how individual frameworks of knowledge are constructed, and this may well require recourse to theories of general cognitive capacities, perhaps of a Piagetian kind. The observation of 'stages' of cognitive functioning, or not, would therefore seem to depend largely upon one's point of view, as Entwistle (1979) has suggested. By concentrating on the similarities in task performance, one may well be led to a belief in the existence of 'stages' in general cognitive functioning. By focussing on the dissimilarities, however, such a construct appears to be less useful and one must look to the operation of task-specific structures.

From a consideration of all these issues, it seems reasonable to suggest, therefore, a picture of conceptual acquisition by which the child's understanding in a given area depends upon the attainment of a certain developmental cognitive level, which in turn permits the construction of spheres of ability or frameworks of knowledge which themselves require instructional or experiential intervention for their full realisation. Such a model has, in fact, been put forward by Demetriou and Efklides (1979, 1981). On the basis of an examination of the empirical findings of research related to formal operational thinking, and on the basis of their own (1979) observation of lack of unity of the major aspects of formal thinking in a 'structure d'ensemble', Demetriou and Efklides (1981) suggested the differentiation of two levels of formal thought which they termed 'strategic' and 'tactic' respectively. Attainment of the strategic level provides the individual with a general problem-solving orientation which functions as a necessary but not sufficient condition for the attainment of various tactics which are "demarcated in respect of their logical constitution and their application domain" to form "spheres of formal thought" (Demetriou & Efklides, 1981). These spheres of formal thought which become differentiated out of the general strategic level of formal thought require specific experience (or instruction) for their effective crystallisation, and may have different 'ages' of appearance. Consequently, one can expect the same individual to manifest reasoning classified as 'formal operational' in one sphere, without necessarily functioning at that level in another. At the same time, the attainment of the strategic level which forms a common basis to the various spheres of formal thought would imply some degree of communality between performance in the different spheres, thus accounting for the reasonably high correlations between tasks representing different spheres observed in commonly-quoted correlation studies, such as those of Bart (1971) and Lovell (1961a) (see also Shayer, 1979).

The findings obtained from the present investigation may well have relevance for a model of this kind. Consideration of this question requires a closer and more careful examination of the areas of difficulty revealed by the research and in particular of their relative levels of difficulty. Certainly, one moral of the observations noted in this study seems to be that in the 'real world', attempts to classify levels of reasoning without regard to the framework of reference or domain of knowledge within which an individual is working may well lead to distortions of description. Another might be that sole concern with the 'frameworks of knowledge' that an individual constructs, to the extent of denial of the operation of more general and maturation-linked reasoning processes, is similarly unwise. The theory of cognition capable of explaining an individual's performance within a given field will ultimately be that which succeeds in reconciling both these perspectives.

Limitations of the Present Research

As mentioned earlier in this chapter, the areas of difficulty established by the present research are those apparently experienced by children making the particular errors chosen for study. Whether other children who are more successful in algebra also experience similar difficulties, but are able by some means to overcome these, has not been established. It may be that the 'more successful' children have, as a result of their earlier experience, built up a more appropriate framework of knowledge in mathematics, so that these issues are at no stage problematic for them. Observations from interviews held after the study reported here, with a small number of children from the top streams of a girls' selective school who were designated as 'mathematically able' by their teachers, indicate that this latter situation may be the case. However, the matter needs much fuller investigation.

Other limitations of the research arise from the relatively restricted range of errors studied. Inclusion of a wider variety of errors would presumably reveal additional areas of difficulty, thereby leading to a more complete picture of the kinds of conceptual problems children may face when meeting algebra. In addition, the difficulties identified in this study may be expected to lead to predictable errors in other areas of algebra than those studied here, but this has likewise not been investigated.

A further limitation derives from the number and range of items used as indices of the various areas of difficulty delineated by the present research. In some cases (e.g. conjoining in algebraic addition) the number of items was relatively large, but in others (e.g. the interpretation of letters) the number was very small. The difficulty in designing items appropriate to this latter aspect has already been mentioned (see the section on 'The Interpretation of Letters' earlier in this chapter); nevertheless, the development of a more substantial set of items relevant to this aspect would permit a more detailed exploration of this particular issue.

One final limitation relates to the fact that during the class teaching (other teachers) phase of the research, it was not possible for the researcher to observe the classes as they were taught. This restricts the usefulness of this phase of the research in terms of the extra information which might have been obtained from such observation (although one or two of the teachers involved did submit fairly substantial comments based on their own observations). It also means that it was not possible to monitor directly each teacher's interpretation and method of implementation of the teaching programme (although again some indication of this was obtained from subsequent discussion with the teachers or from their written comments). In so far as the between-class differences obtained during this phase in general reflected those found during the researcher-taught phase, this may not have been important. Nevertheless, the possibility remains that some of the difference observed between classes of different age or ability level (such as that relating to children's level of letter interpretation, for instance) may have been due to differences in teaching emphasis on the part of the individual teachers concerned. The questions arising from a consideration of these limitations suggest some useful ideas for further investigation.

Summary

The findings of the present research indicate that children's errors in algebra arise both as the consequence of falsely generalising from notions already established in arithmetic, and as the result of not having appropriate arithmetic structures from which to generalise when such generalisation is in fact required. In addition, the possible operation of a cognitive growth process by which the assimilation of some concepts and procedures awaits the development of particular cognitive structures which take time (and require a particular level of maturation?) to be established has also been indicated. Analysis of the nature of the difficulties which children have been observed to experience suggests a picture of conceptual growth which is generally not inconsistent with Piaget's description of the development from concrete to formal operational thinking. The observation of apparently maturation-linked factors with respect to conceptual notions such as that of generalised number argues for a Piagetian-like picture of a general cognitive growth pattern which is common to different individuals, as does the observation of marked similarity in 'child-method' approaches across individuals and tasks. The general description of the features of these child methods (see Chapter 6) is also not inconsistent with the Piagetian description of concrete operationality. However, the ease with which improvement in performance in other conceptual areas was obtained argues against the association of those abilities with as yet unattained cognitive stage structures, and supports the notion of the relevance of appropriate frameworks of knowledge for children's algebraic functioning. The observation that the (pretest) incidence of errors studied was of a similar order in groups of children from first year (age 12) to fourth year (age 15) suggests that the fact of cognitive development (in the Piagetian sense) does not in iteself ensure the growth of understanding in these particular areas (assuming that such cognitive development would be expected to be proceecing over this time span), and indicates that consideration must also be given to the kinds of knowledge framework that children are constructing during this period.

CHAPTER 10: IMPLICATIONS OF THE RESEARCH

As will be clear from the discussion of the previous chapter, the findings of the present research may be seen to have implications for theory and for further research, as well as for the teaching of algebra and indeed for mathematics in general. This chapter outlines what some of these implications may be.

Implications for Teaching

The Meaning of Letters in Algebra

As in the case of the research cited earlier, this project has drawn attention to the difficulties which many children have concerning the meaning of letters. Whilst some of this difficulty may have a more fundamental conceptual basis, nevertheless the research has also revealed sources of misconception which may be effectively addressed by the teaching process. In this regard the following recommendations are suggested:

1. Since many children appear to be predisposed to the idea of letters as specific unknown values, it may be useful to adopt the generalised number interpretation of letters from the time that letters are first introduced. This can, of course, be done via the medium of a 'mathematics machine' context as used in the present study. However, the same approach can also be used even in the more conventional approach to algebra. For example, in dealing with simple equations (e.g. $x+5 = 8$), the assumption is usually made by child and teacher alike that the x represents only one value. However, it is just as correct, and algebraically perhaps more

useful, to view the x as taking values from a whole range of values, only one of which (in this case) makes the expression true.

This approach also requires a careful assessment of the 'general' nature of the proposed letter usage. Many teaching schemes, for example, utilise an approach in elementary algebra in which the child investigates a series of particular numerical instances of a given relationship (such as the area of different sized rectangles), writing down each individual numerical relationship before 'generalising' the rule in the form of an algebraic statement. However, from the child's point of view, this algebraic statement can just as easily be regarded as indicating what will be true <u>in the next specific instance,</u> or as representing in shorthand form what the child has just done in a given specific case, rather than representing a summary statement of what is true in all conceivable instances of that kind. Consequently this usage may still be interpreted by the child in a 'specific unknown' rather than 'generalised number' sense.

2. Since many children confuse the arithmetic and algebraic usage of literal expressions, it may be useful to make this distinction explicit by discussing, for example, the alternative meanings of terms such as 3m ('3 times m' or '3 metres'). This discussion may usefully include consideration of the meaning of such symbols as the equals sign.

3. Several work schemes use algebraic initialization such

as 'v = the number of vertices' as a mnemonic device. However, since some children do not always appear to make a distinction between 'v for vertices' and 'v for the number of vertices', such usage may be best avoided, at least initially.

4. Children need to be encouraged to reflect upon the meaning of the mathematical expressions they meet. This is essential to an appreciation of the need for rigor in symbolising different mathematical operations (such as division and subtraction expressions) and so must form a part of any attempts to help children's understanding of the formalization of mathematical procedures (see below). It is also useful to the child's handling of algebraic simplification exercises of the kind '2a+5b+a'. The consistently successful handling of examples of this type can only proceed from a awareness of the letters as representing possibly different numerical values. (Contrary to expectation, the 'apples and bananas' explanation for the non-combinativity of terms such as 2a and 5b is not to be relied upon: in the present research children giving this explanation were just as likely to produce an incorrect answer as a correct one.)

The Formalization of Method.

The research reported here suggests that a fuller understanding of algebra (and perhaps of mathematics in general) requires that attention be paid to the kind of methods that children use, and to ways of assisting children to become aware of the uses and limitations of different kinds of procedure. Attention is also required to the ways in which these procedures can be symbolised. As a first step, this requires the teacher to recognise that the child may be using informal methods rather than the taught procedures, and to investigate the kinds of informal method which are so used, drawing the child's attention to the nature of these methods, their usefulness, and their limitations. Only when children become aware of the limitations of their own methods, it is suggested, will they be prepared to contemplate the value of the more formal methods which the teacher is attempting to teach. Ways in which this might be achieved may be suggested by reference to other areas of algebra besides those included in the present research. For example, despite the fact that teachers may use simple algebraic equations of the kind $x+5 = 8$ as a medium by which to teach the formal equation-solving process, the children's method may be to proceed to the solution of such equations by inspection, 'guess and test', or the application of known number bonds. While these strategies can bring the children success on these easy items, however, they cannot be effectively utilized in the case of more complex equations, where a correct solution usually requires the application of the formal taught procedure. Hence as long as children do obtain success by the use of their own informal methods, they will be unlikely to attend to the (perhaps more obscure) procedures being developed by the teacher, with inevitable consequences when the more complex equations are eventually met. To remedy this situation, the child's method and its usefulness in solving easy equations must be recognised and the possibility of applying these procedures to complex equations must be assessed, by the simple process of attempting to use them in such an instance. For example, having discussed the usefulness and ease of application of the child's trial and error method in the case of simple equations of the kind '$x+5 = 8$', the teacher might present a more complex example such as:

$$\frac{3x+8}{5} = 15-\frac{2x}{3}$$

with a view to discussing the possible application of the child's procedure to equations of this kind.

The recognition of the limitations of the child's own informal methods in this regard will alert the child to the need to contemplate more powerful procedures. The way to a consideration of the formal methods will now be open; it will remain for the teacher to determine the most effective ways of assisting the child to deal with whatever conceptual complexities those methods may embody. The same process, of course, may be used in alerting children to the desirability of formalizing procedures in other areas of mathematics, and particularly in those aspects of arithmetic which form the basis for later algebraic representation.

This is not to imply that children's informal procedures are of no value. Indeed, the formal procedures taught can only acquire meaning for the child if they are, at least initially, related to those more intuitable methods which the child already understands. Secondly, of course, the informal procedures are of value in their own right where the solution of simple problems is concerned, as in the case of the '$x+5 = 8$' equation discussed earlier. The problem for teaching lies in the need to help children acquire facility with both kinds of approach, and to perceive the relationship between them. The jump from the intuitable to the more formal procedures is likely to present difficulties for children, and is consequently a question which requires careful consideration in order to develop processes by which the transition can be meaningfully brought about.

This focus on the nature of the procedures required to solve a given class of problems may also help to shift children's attention from an over-concern with the final (numerical) answer. This emphasis on the final answer pervades much of children's early experience in arithmetic, and indeed proceeds into later mathematical work. In introducing work on equations, for example, many teachers define the activity to be engaged upon as that of 'finding "x"', rather than that of developing general equation-solving procedures which can be used to find the unknown value in a whole range of problems. A de-emphasis on the requirement for a final numerical answer has been suggested to be a pre-requisite for children's acceptance of the unclosed expression as a legitimate 'answer'. Certainly the approach taken by the teaching programme developed in the present study, by which this de-emphasis was attempted, was effective in promoting such acceptance.

One other point may be appropriate. In terms of the strategies listed in Chapter 2 as being potentially useful in the study of children's mathematical behaviour, this thesis has been primarily concerned with two, namely the 'informal intuitive strategy' and the child's proceeding via the 'recognition of formal structure'. It was mentioned earlier that the involvement of trial and error or pattern-seeking procedures was not anticipated in view of the nature of the particular items being investigated. Indeed, in general these strategies were not observed, except for such instances as the child's tendency to complete as unclosed 'perimeter' shape by symmetry, which was noted in the early phase of interviews, and the child's replacement of a letter by the number signifying that letter's position in the alphabet, both of which may be regarded as procedures exemplifying a pattern-seeking strategy. However, the 'search for an algorithm' strategy, by which was meant the utilization of a rote-learned procedure, might reasonably have been expected to occur, and yet was rarely observed. One exception to this was, of course, the repeating of the 'apples and bananas' type of explanation for the non-combinativity of unlike algebraic terms. The teaching of this rule had not been completely successful, however, since children repeating it seemed just as likely to give the wrong answer as the correct one. This suggests that purely algorithmic or 'rule' teaching which is not based upon some prior understanding of the child's, may not be assimilated and hence not used correctly by the child at a later date when required. Certainly there was little evidence in the present study of children's rote repetition of what the teacher or textbook had said, seemingly indicating that this approach was not one often used by these children.

Notation and Convention

Finally, the problems which many children have with symbolisation must not be underestimated. Some children do not readily assimilate the meaning of abstract representations, and care must be taken to ensure that attention is paid to the established processes by which meaning can be so attached. The value of 'operationalising' symbolic representation and of introducing intermediary signs has already been touched upon (Chapter 9). Other procedures include 'practice' in the sense of repeated use of a given notational representation; 'discrimination' in terms of the consideration of both positive and negative instances of the symbol's referents; and 'reinforcement' by repeated interpretation of the symbol's meaning. The consideration of non-examples and in particular the distinction between a given form of notation and one likely to be confused with it is especially important in the process of attaching meaning to symbolic representation (see also Gerace & Mestre, 1982). This is highlighted by the observation that children did not find the 'consolidation' exercises in the teaching programme used in the present research a trivial exercise, and is further indicated by the fact that many children who correctly answered the conjoining items and the first two 'use of brackets' items, nevertheless selected one or more of the incorrect options in the item presenting a range of possible responses for the product of 3 and e+2. This item was particularly interesting in that virtually none of the children gave exactly the same answer twice (or more) over the three tests. Evidently children may find discriminating between correct and incorrect answers far harder than producing a correct response. It is suggested that this indicates that different levels of understanding are required to correctly handle the two kinds of problem, and that the obtaining of a correct response may only give a partial picture of the child's understanding.

General Teaching Principles

The work in the present study has also permitted the value of various general principles in mathematics teaching to be reiterated. In particular, interest in the present research has focussed upon:

1. the need to anticipate areas of difficulty and understand their likely bases; this can be done most effectively by -
2. the analysis of common errors made by children in the topic under consideration, which in turn may be done most efficiently by means of -
3. discussion with children, both in an individual and

class context; in this regard it is important to pursue the reasons for common wrong answers, rather than merely concentrating on the correct answer;

4. the value of setting teaching within a wider context which gives meaning to the concept or procedure to be taught (for example, the teaching of procedures within the context of programming a computer, or the teaching of algebraic manipulations within the context of solving equations rather than as an exercise in its own right);
5. the importance of giving attention to the <u>procedures</u> which children use, even when getting the correct answer;
6. the need to pay deliberate attention to the problem of the teaching of notation; and
7. the need to help children recognise and appreciate the equivalence of different conceptual viewpoints, and to be able to shift readily from one to another as required (as in appreciating that an expression such as 'n+3' can mean both an instruction to operate on the two elements, and the result of performing that operation).

Other general principles for the teaching of mathematics, and which were utilized in the present study, have been indicated in Chapter 7.

Implications for Further Research

The Use of Informal Methods in Mathematics

Perhaps the point of greatest interest arising from the present research is the indication that many children do not seem to have formal representations of the methods they use in solving mathematical problems, and indeed that they may not use the formal 'taught' methods, but may instead use more informal procedures of their own. That these informal methods can be applied successfully in the case of simple examples may be readily demonstrated; many adults doubtless also use the same procedures (see Committee of Inquiry into the Teaching of Mathematics in Schools, 1982). However, the solution of more complex problems, and certainly the extension of mathematical knowledge, almost certainly requires the formalization of general procedures which can be applied to a much wider class of problems. It is suggested that this formalization requires the recognition (on the part of both teacher and learner) that the construction of such procedures should constitute a significant part of mathematical activity. This issue merits further investigation, particularly with regard to the following questions:

1. How widespread is the use of informal methods among children at both primary and secondary school level?
2. How do these informal methods develop, and why are they so resistant to change?
3. Why do many children fail to assimilate the formal taught procedures?
4. If it is desired that children develop an awareness and understanding of the formal methods, how can this best be achieved?
5. Where children who are low attainers in mathematics are concerned, it may be considered that the development of a formal understanding of mathematics is less appropriate. In this case, can the informal or 'child-methods' be usefully developed and extended? This requires that the nature of such methods be fully explored.

With reference to these questions a new project based at Chelsea College and funded by the Social Science Research Council is investigating the development in young children (aged 8 to 13) of various formal and informal methods in mathematics (see CMF Research Proposal, 1982). In particular the project aims to study the nature of the interaction between the child and the knowledge presented by the teacher and to investigate the kinds of procedures which are subsequently constructed by the child. The general premise of the research is that many children develop successful commonsense methods at an early stage in their mathematical experience, thereby avoiding the development of a formal mathematical framework. In the light of its findings, the project will attempt to suggest means by which the transition from the informal to the more formal 'taught' procedures may be effected more efficiently.

At the same time, the value of these informal procedures must not be overlooked, nor indeed must their appropriateness to the more simple mathematical problems which the child (and adult) may meet. The point is not that it is desired to move the child entirely away from these methods, but rather that the child be helped to perceive the relationship between formal and informal procedures and to choose that approach which is more appropriate to the particular problem in hand. As indicated in point '5' above, it may be that the 'informal approach' can be usefully developed and extended for, say, low-attaining children in mathematics, without losing the possibility for later transition to the more formal procedures should this be subsequently desired. A description of the features of the informal procedures used by children has been suggested by the SESM research (see Chapter 6) and is reproduced below:

Informal methods used by children are characterised by being -

1. intuitive, i.e. based upon instinctive knowledge: not systematically reflected upon and not checked for consistency within a general framework;
2. primitive, i.e. tied closely to early experiences in mathematics;
3. context-bound, i.e. elicited by the features of the particular problem;
4. indicative of little or no formal symbolised method;
5. based largely upon the operations of counting, adding and combining;
6. worked almost entirely within the system of whole numbers (and halves).

The investigation of the validity of this description across a wider problem-base may usefully be pursued, with the aim of using the description in order to develop other procedures of similar kind which may form the basis for instruction with selected groups of children.

<u>The Nature of Cognition and Children's Understanding of Algebra</u>

Other areas of investigation suggested by the present research relate to the question of the nature of children's cognition and the relevance of this for questions of mathematical learning. The importance of context for children's problem-solving activity and the difficulty which children appear to have concerning the notion of generalised number, are both predicted by the Piagetian theory. At the same time, the relative ease in the assimilation of the notion of lack of closure, and the improvement in formalizing mathematical method demonstrated in the present study, would seem to indicate some inconsistency mitigating against the unqualified acceptance of the 'unified stage' view of cognition which characterises the Piagetian formulation. Indeed, this view has been the source of much criticism (see, for example, Brown and Desforges, 1979), and Collis himself has more recently proposed that the nature of the content or activity be taken into account, suggesting in preference to 'cognitive stages' a taxonomy of 'observed learning outcomes' - levels of performance based on Piagetian-like descriptors but which are topic or task specific (see Biggs and Collis, 1982). In this regard, the implications of the framework of knowledge or reference within which a child works for that child's level of performance in a given domain, and in particular for the kinds of errors which are made, require further investigation. Nevertheless, the observed similarities in the nature of the informal methods used by different children, as well as the points concerning context and the generalized nature of algebraic representation outlined above, suggest some generality in cognition which requires explanation. The mechanisms by which these various task-specific and task-independent abilities are coordinated remain to be elaborated.

In approaching this task, it is suggested from the present research that the notion of 'complexity' in describing the problems which are set children, and upon which inferences concerning level of cognitive functioning may be derived, may required careful definition. The psychological complexitities of a task may depend upon such aspects as (i) the kind of procedure or strategy that the child uses, (ii) the child's understanding of the 'rules of the game', (iii) the child's specific knowledge relating to that task and the degree of automaticity which such knowledge has acquired, (iv) the child's information-processing capacity, and (v) the child's general cognitive level. Any discussion of psychological complexity which does not take (at least) these factors into account must be necessarily incomplete.

Misconceptions in Algebra

This project has suggested some of the areas of difficulty underlying children's misconceptions and error in elementary algebra; there may well be others. This research has been concerned with highly specific errors relating to a narrow range of algebraic items. Consideration of other types of problems and errors in algebra may be expected to reveal other sources of misconception. For example, the move from arithmetic to algebra, besides necessitating modifications in notational representation and conceptual and procedural viewpoints may also require a change in nature of the problem-solving process. Whilst more applicable to later work in algebra than that which forms the basis of this study, there are some indications from the present research that successful handling of algebra even at the generalised arithmetic level may accompany the beginnings of a change in problem-solving orientation. Many algebraic directives, such as 'solve', 'simplify', 'factorise', etc. describe the features of the desired result rather than prescribing a single specific procedure to execute, as is more commonly the case in arithmetic (see Matz, 1980), and this difference in specification may require a conceptually different problem-solving approach. Petitto (1979), for example, has discussed the relevance for algebra of the distinction between wholistic and algorithmic problem-solving styles (after Wertheimer, see Petitto, 1979), while Davis (1975b) has noted that children can often do 'small steps' in mathematical problem-solving while having no overview of the total procedure, suggesting that these two kinds of mathematical thinking, which he terms 'algorithmic' and 'conceptual', may function differently and that a lack of emphasis during teaching on conceptual thinking may underlie much of the difficulty which children have in algebra, and indeed in mathematics generally. Children's apparent inability to view algebraic problems globally has also been suggested by other workers to contribute to children's algebraic difficulties (e.g. Herscovics & Kieran, 1980; Matz, 1980; Wagner, 1981b; Wheeler, 1981). More likely it is the ability to switch from one kind of reasoning to another which contributes to children's ability to perform successfully in this topic. Nevertheless, the fact that algebra may require a different kind of problem-solving style to that appropriate to arithmetic may indicate another aspect in which working within an arithmetic framework may lead to error when dealing with algebra, and is therefore one which merits investigation.

There are doubtless other distinctions between arithmetic and algebra-as-generalised-arithmetic which can be drawn and which may reveal potential psychological complexities. Whether these can be easily surmounted by the learner must depend upon the latter's psychological characteristics which must consequently be investigated by studying the student rather than the topic per se.

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APPENDICES

APPENDIX 1A

Examples of CSMS Algebra Items

Item 5

If $e + f = 8$

$e + f + g = \ldots\ldots\ldots$

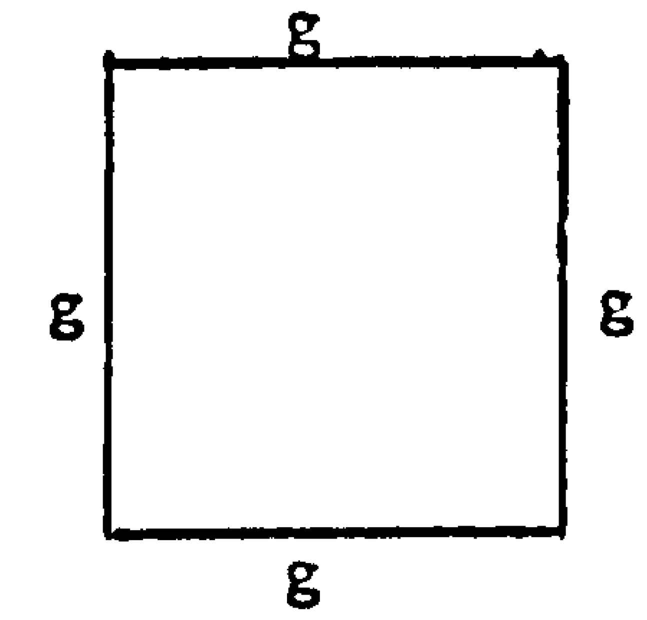
Item 6

What can you say about a if $a + 5 = 8$

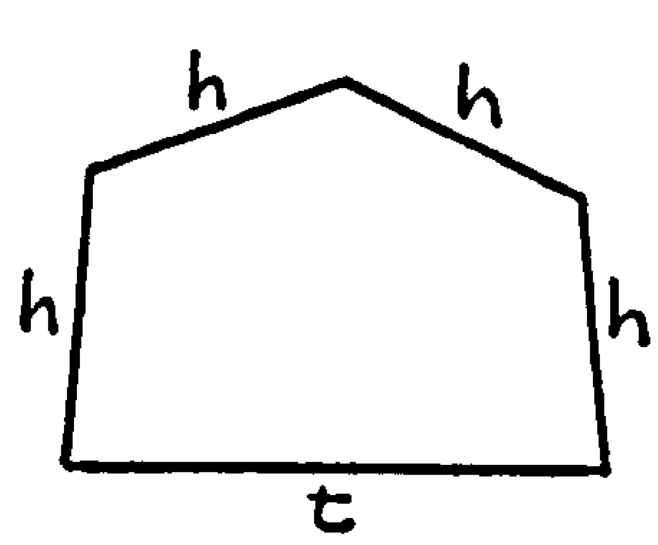
Item 9

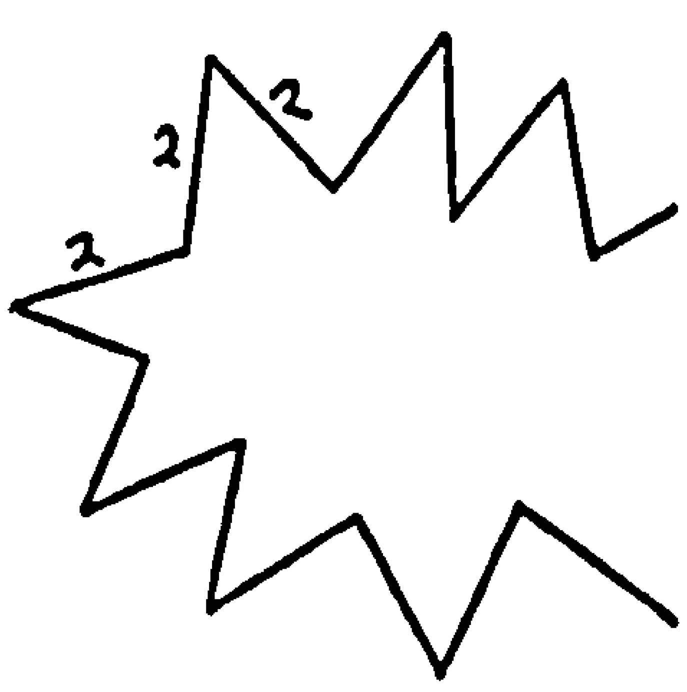
This square has sides of length g.

So, for its perimeter, we can write $p = 4g$



What can you write for the perimeter of each of these shapes?





Part of this figure is not shown. There are n sides altogether, all of length 2.

Item 12

If John has J marbles and Peter has P marbles, what could you write for the number of marbles they have altogether?

Item 18

When are the following true - always, never, or sometimes?

Underline the correct answer:

$A + B + C = C + A + B$ Always, Never. Sometimes, when ......

$L + M + N = L + P + N$ Always, Never. Sometimes, when ......

APPENDIX 1B

Levels of Understanding in Algebra Identified by CSMS

(taken from Küchemann, 1981, pp 112-116

Level 1 The items at this level are purely numerical as in (1) below or they have a simple structure and can be solved by using the letters as objects (2), by evaluating the letter (3), or by not using the letters at all (4). For more complex items children at this level tended to give answers like 8ab instead of 3a + 5b. In the case of items that required specific unknowns these children were likely to evaluate the letter (5), or not use the letter at all (6).

(1) What can you write for the area of a rectangle measuring 6 by 10 units.

(2) Write more simply 2a + 5a.

(3) Find a if a+5 = 8.

(4) Evaluate a+b+2 if a+b = 43.

(5) 12 given instead of 8+g as the expression for e+f+g given e+f = 8.

(6) 7 given instead of 3n+4 in answer to 'add 4 to 3n'.

Level 2 The clear difference between these items and those at Level 1 is their increased complexity, though the letters still only have to be evaluated, or used as object. Children at this level could still not consistently cope with specific unknowns, generalised numbers or variables.

It might be argued that the advance made at this level is due simply to an increased familiarity with algebraic notation. However, the children who coped with Level 2 items also performed more successfully on the test as a whole. More significantly, their average raw score on the Calvert DH IQ test which can be regarded as an indicator of ability external to the algebra test, was also higher. This suggests that the use of correct syntax is, at least in part, conceptual.

The improvement in the Level 2 answers over those of Level 1 can also be seen as a first indication (which is much more fully realised at Level 3) of a willingness to accept answers which are to some extent 'incomplete' or ambiguous, which Collis describes as 'acceptance of lack of closure'. Thus, for example, many of the children who wrote 8ab as a simplification of 2a + 5b + a, (of whom about three-quarters were at Level 1) may well have known how to write 3a + 5b but preferred their answer because it looked more 'proper' and complete.

Level 3 The major advance made by children at this level is that they can use letters as specific unknowns, though only when the item-structure is simple. These children are able to regard answers like 8 + g, 3n + 4, p = 2n as meaningful, even though the letters represent numbers and not objects, and despite the lack of closure of the answers.

Level 4 At this level children can cope with items that require specific unknowns and which have a complex structure, as in (1) below. They can also cope with items like (2) which require, at a minimum, that the letters are regarded as specific unknowns, but where there is a stong temptation to treat them as objects (e.g. cakes and buns). Other items involve generalised numbers, or the necessity to realise that a set x can equally well be represented by an expression like 5x and that, furthermore, this results in the transformation 'divide by 5' and not 'multiply by 5' on the values of x (e.g. if $(x+1)^3 + x = 349$ when $x = 6$, what value of x makes $(5x+1)^3 + 5x = 349$ true?).

(1) Write more simply (a-b)+b.

(2) What is the meaning of 4c + 3b (where cakes cost c pence each and buns cost b pence and 4 cakes and 3 buns are purchased.

Note: Each 'level' comprises a set of items which are grouped in terms of difficulty and mathematical cohesiveness. These groups of items are scalable in the sense that a child's success on any level (defined as correctly solving two-thirds or more of the items at that level) entails success, as defined, on all easier levels.

APPENDIX 2A

SESM Algebra Marking Scheme

The marking scheme used for the CSMS algebra screening test used in the present research was that used by the CSMS team (given in Hart, 1980b, pp.178-180). For purposes of selecting children for interview, however, certain categories of error answer identified by CSMS were combined.

The list of items used as a basis for selecting children for interview, together with the error answer categories adopted by the present research, are given here. A brief rationale for the combining of CSMS error categories, in those cases where this was done, is also given.

CSMS Item No.	Abridged Item	CSMS Error Categories	Error Answers	SESM Category
4(ii)	Add 4 to 3n	3, 5	3n4, 7n 43n	Letter as object
		6	7, 12	Letter ignored
4(iii)	Multiply n+5 by 4	3	4n5, 20n n54	Letter as object
		6	20, 9	Letter ignored
5(iii)	e+f = 8	3	8g	Letter as object
	e+f+g = ?	6, 8	9, 12	Letter ignored
		7	15, 8	Alphabetic
7(iv)	Area 5 by e+2	3, 5	532, 325 10e, 7e	Letter as object
		6	10, 7	Letter ignored
9(ii)	Perimeter 4h+t	3, 5	hhhht, 4ht 5ht	Letter as object
9(iii)	Perimeter 2u + 16	3	uu556, 2u16 2u556, uu16	Letter as object
9(iv)	Perimeter 2n	2, 3	n2; n,2	Letter as object
		8	32 - 42	Letter ignored
		7	28, p	Alphabetic
12	J + P marbles	3	JP; J,P	Letter as object
		7	26	Alphabetic
13(ii)	2a + 5b	3, 5	7ab, 8ab	Letter as object
13(iv)	2a + 5b + a	3, 5	3a5b, 8ab 8aba, 7aba	Letter as object
13(viii)	3a - b+a	3, 5	4ab, 5ab 2ab	Letter as object
14	r = s+t	-	18	Alphabetic
	r+s+t = 30			
	r = ?	8	10	Letter ignored
15(ii)	k - 3 diagonals	4	k3, 3k	Letter as object
		6	3, -3	Letter ignored
		7	h, i, 8, 9	Alphabetic
16	c+d = 10	6	single value	Letter as specific number
	c<d			
	c = ?			
18(ii)	L+M+N	9	'Never'	Letter as specific number
20	Cakes and buns 4c + 3b	8	4 cakes and 3 buns	Letter as object

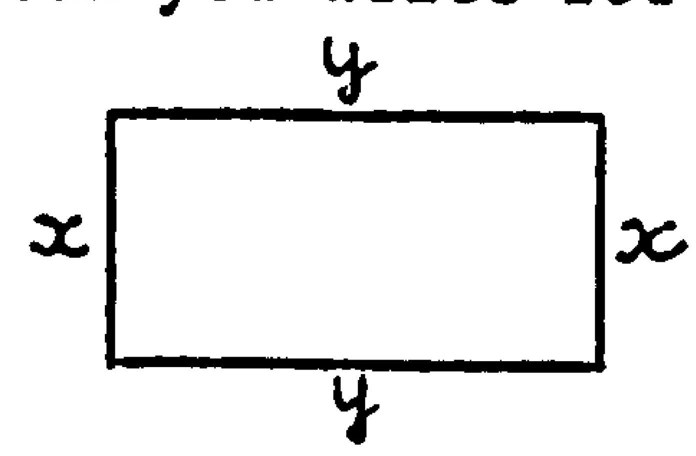
Note on justification for combining CSMS error categories:

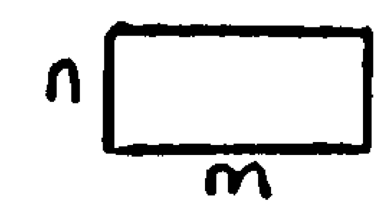
Items 4(ii), 7(iv), 9(ii) 13(ii), 13(iv), 13(viii) (Combining CSMS categories 3 and 5)	Here the significant feature was considered to be the conjoining of letters and numbers into a single term, e.g. 2a + 5b = 7ab <u>or</u> 8ab.
5(iii) (Combining CSMS categories 6 and 8)	Here the significant feature was taken to be the omission of the letter, rather than the actual numerical value given.
9(iv) (Combining CSMS categories 2 and 3)	Here 'n2' was considered ambiguous and may represent a conjoining of elements rather than an unconventional ordering of the product term.

APPENDIX 2 B

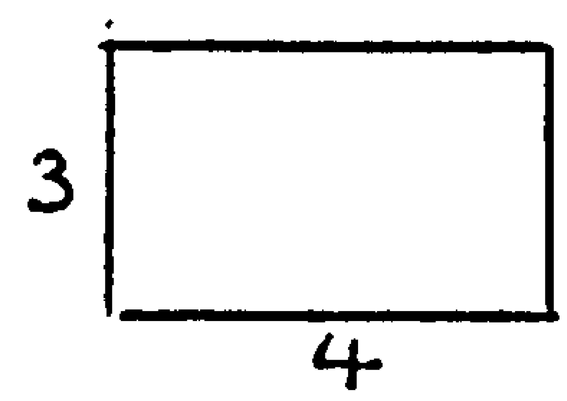
Interview Items : Phase One

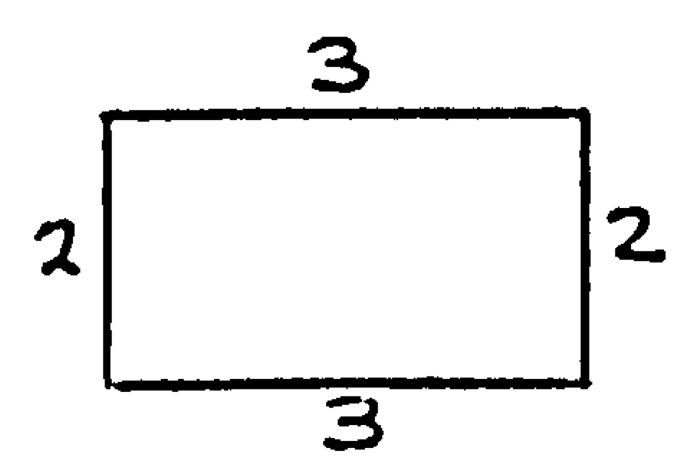
A. What can you write for the area of the rectangle:

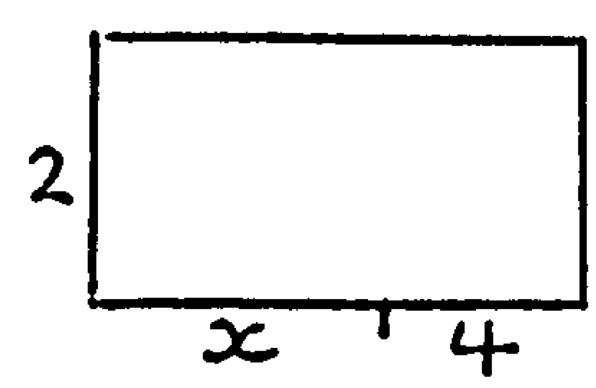
1. 

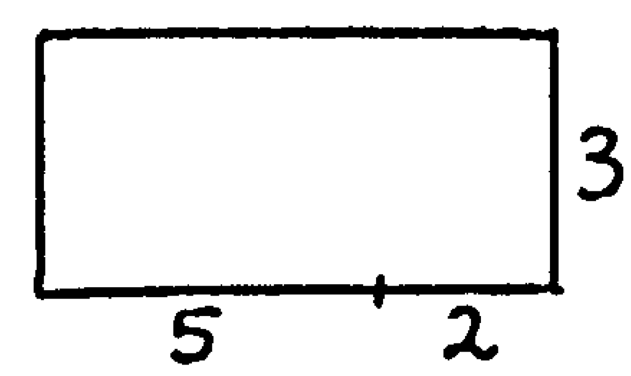
(Amended from 

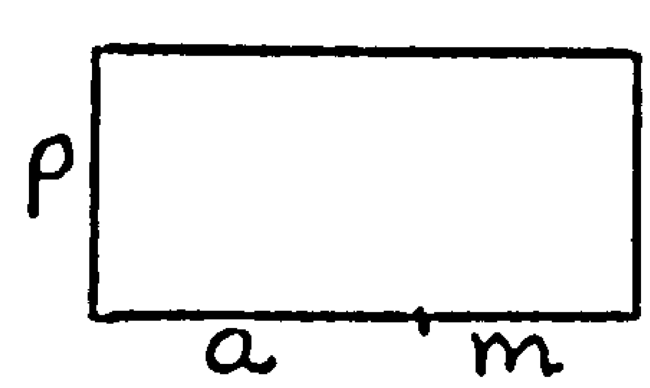
to ascertain influence of additional information.)

2. 

3. 

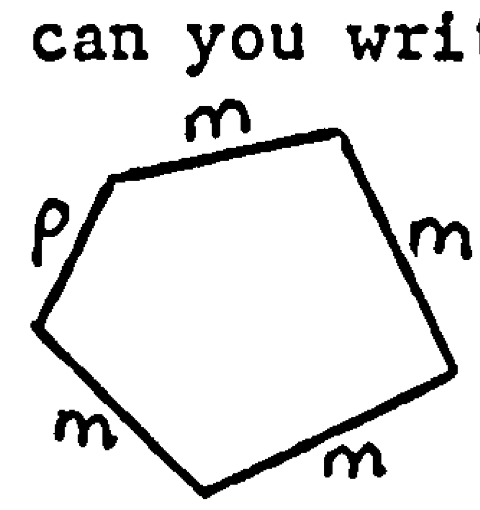
4. 

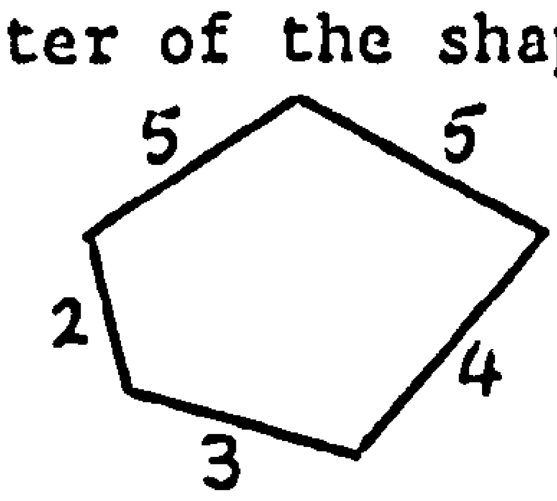
5. 

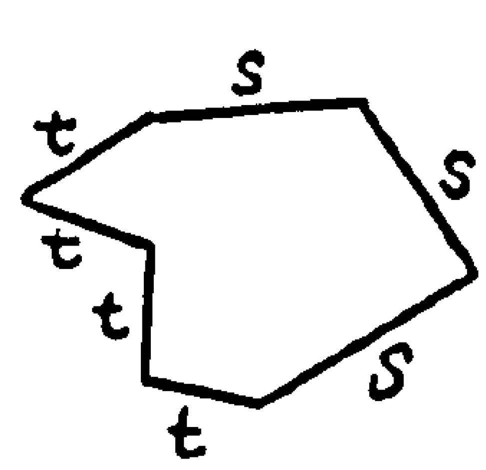
6. 

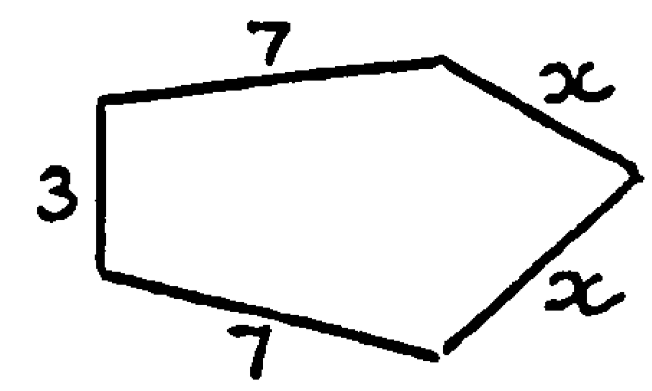
{ Extension item to (4) to ascertain effect of absence of numerals. }

B. What can you write for the perimeter of the shape:

1. 

2. 

3. 

4. 

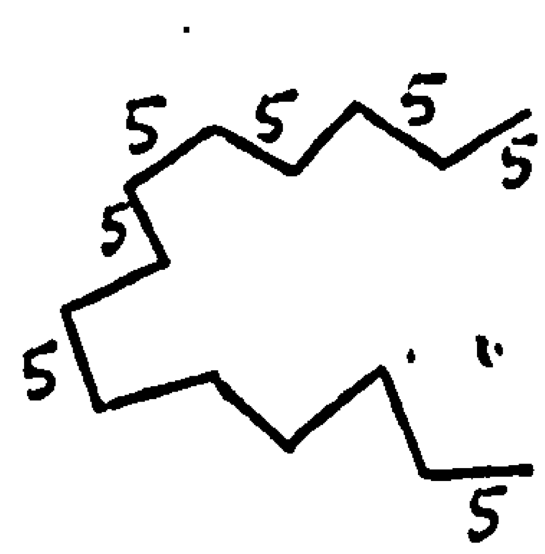
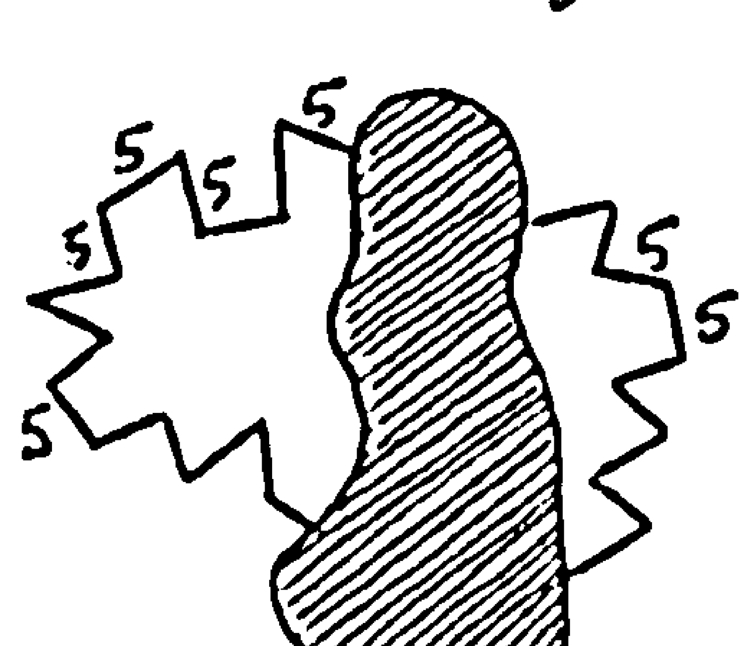
5. 

Figure has p sides altogether.
Each side is 5 units long.

6. 

(Amended from (5) to avoid possible confusions due to diagram).

C. West Ham scored x goals and Manchester United scored y goals. What can you write for the number of goals scored altogether?

D. $x + y + z = x + p + z$

Is this true: always / never / sometimes, when ....

(Amended from $L + M + N = L + P + N$ to check if any difference in interpretation due to use of upper case letters.)

E. 1. Add 3 on to $5y$ 2. Multiply $k + 2$ by 3

3. Write more simply if possible:

a) $2x + 5y$ c) $5q - 3w$

b) $3x + 8y + 2x$ d) $5p - t + 2p$

F. What can you say about k if $k + m = 12$ and k is less than m?

Note: a. Each child received a selection from the items shown; no child was presented with every item.

b. Numerical items were used where the child was unable to proceed with an algebraic item, in order to ascertain how the child would handle the arithmetic case.

APPENDIX 2 C

Interview Schedule Used as a Basis for Interviews In Phase One and Two

	Process	Question Asked (Example)
1.	Reading	Please read the question out loud.
2.	Comprehension	What is the question asking you to do?
	Interpretation	What does .... mean?
3.	Strategy Selection Skills Selection	How will you do this question? Why?
4.	Process	Work out the question and tell me what you are doing as you go.
5.	Memory	(Check for recollection of intermediate step).
6.	Encoding	Now write down the answer.
7.	Consolidation	What does the answer mean?
8	Verification	Is there any way you can check to make sure your answer is right? Suppose I said I didn't believe you, could you prove to me that your answer is right?
9.	'Conflict'	Suppose I put 3 for x ...... Another student said ......
10	Similarity	Can you make up a question like that one? Which of these questions is like that one? Why? Which is different? Why?
11	Generalisation	(Solving problem of more complex or more abstract nature, or one set in different context).

APPENDIX 3

Samples of exercises from each worksheet used in the small-scale teaching experiments (see Chapter 4).

1. Introduction

Worksheet No. 1

Write down the instructions you would have to give the machine if you wanted it to do the following problems. Write the instructions both ways if possible. CHECK that you have written what you mean.

1. Add 19 and 38.
6. Add 17 and 34 and multiply the answer by 6.
11. Divide 4 by 10.
15. Multiply 7 by 39, add 16, and then take away 42.

Now use your calculator to work out each answer and write your answers in the 'print-out pad' box.

2. Number Operations I

Worksheet No. 2

Write down the instructions you would have to give the machine if you wanted it to do the following problems. Write the instructions both ways if possible. CHECK that you have written what you mean.

1. I drive 67 kilometres before lunch, and 98 kilometres after. How many kilometres do I drive altogether?
10. A crate of soft drink holds 72 cans. I order 52 crates. How many cans of soft drink will I get?
11. A fat man wants to lose 57 lbs. weight. So far he's lost 29 lbs. How many more lbs. must he lose?
15. Make up a problem, and then write down the machine instructions for it.

Now use your calculator to work out each answer and write your answers in the 'print-out pad' box.

3. Generalisation I

Worksheet No. 3

Write instructions for the following. Write the instructions in as many different ways as you can. CHECK that you have written what you mean.

1. Add 8 to any number I give.
7. Subtract 9 from any number I give, and then multiply by 15.
12. Take any number I give away from 100.
15. Add any number I give to 23 and then double the answer.

(Programme then requires consideration of the 'general answer' as well as specific answers obtained by particular replacement values).

Worksheet No. 4

Write instructions for the following. Write the instructions in as many different ways as you can. CHECK that you have written what you mean.

1. Add any number I give to itself.
6. Think of a number, multiply it by 7, and then take away the number you first thought of.
8. Multiply together any two numbers I give.
15. Multiply any number I give by itself, and then take away another number.

(Programme requires consideration of 'general' and of specific answers obtained by particular replacement values.)

Worksheet No. 5

Write down the print-out answer:

	Instruction Pad	Print-out Pad
1.	a+6 a=4 a=100 a=2½	
2.	20-n n=10 n=0 n=11.5	

4. Generalisation II

Worksheet No. 6

Write instructions for the following. Write the instructions in as many different ways as you can. CHECK that you have written what you mean.

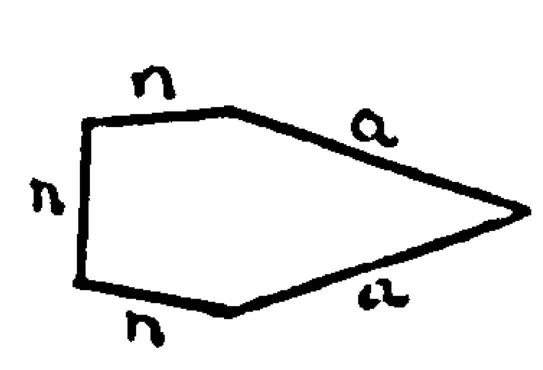
1. Find the area of any rectangle.
4. Find the perimeter of any triangle.
9. I have b pupils. I'm going to give them y pencils each. How many pencils will I need?
15. Find the perimeter of a shape which has x sides which are all 3cm. long.

5. Notation

Worksheet No. 7

Find the perimeter of each shape:

1. 4-sided figure, all the sides must be the same length.
6. 5-sided figure, all the sides can be different lengths.
10. A figure which has 20 sides altogether, 13 of which are all one length, and 7 sides are all another length.
11. Shape:



Worksheet No. 8

What do these expressions mean? Write your answer as many ways as you can.

1. (3xa) + (4xa).
6. (10xa) - (6xa).
8. (2xt) + (7xm).
9. (8xp) - (5xa).

Worksheet No. 9

What do these expressions mean? Write your answer as many ways as you can.

1. 2a
7. 5t
9. 2p + 7p
17. a+b+a

6. Consolidation

Worksheet No. 10

Some of these answers are wrong. Cross out all the wrong answers. Write the correct answers in the shortest way.

	Instruction Pad	Print-out Pad
1.	Divide 8 by any number I give	x÷8 8÷x
5.	Add together any three numbers I give	x+x+x a+b+c n+n+3 5+2+7
8.	Add p five times	p+5 p+p+p+p+p 5+5+5+5+5 px5

APPENDIX 4

Examples of amendments to worksheets following trial with the material in the small-scale teaching experiments described in Chapter 4. The worksheets amended as indicated below were then used in the class teaching (researcher) phase of the research (see Chapter 5).

1. Introduction

Amendment: The inclusion of non-integral values.

Worksheet No. 1

6. Add 12.3 and 7.01.
12. Multiply 5.7 by 11.32.
15. Divide 4.01 by 10.6.

2. Number Operations

Amendment: The inclusion of non-integral values.

Worksheet No. 2

4. A rocket ship goes 93 miles in 1 second. How far will it go in 17.5 seconds?
12. I've got £42.60. My brother has £17.90 more than me. How much has he got?
14. The length of a rectangle is 15.4cm and its width is 7.6cm. What is the area of the rectangle?

2a. Notation - Brackets

Amendment: The inclusion of a section to point out the importance of brackets in mathematics in general, as well as for the 'mathematics machine'. The order of operations conventions were overridden and the rule introduced that brackets would be needed in order to indicate priority of operation. The purpose of the examples shown below was to highlight the fact that different values can be obtained for the same numerical expression according to the order in which operations are performed, and to show that some means must therefore be used to indicate which order of computation is required.

Worksheet No. 3

Work out the following in as many different ways as you can:

1. $3 \times 2 + 4$.
2. $100 \div 10 \div 5$.
6. $7 + 3 \times 8$
9. $3 + 3 \times 10 - 2$

Put brackets in to show how I got the answers:

1. $3 \times 4 + 2 = 18.$

8. $20 \div 2 + 3 = 13.$

9. $20 \div 2 + 3 = 4.$

10. $4 + 5 \times 8 - 5 = 27.$

Worksheet No. 4

Write down the machine instructions for the following:

1. Add 5 and 7 and then multiply the answer by 3.

4. Add 93 to the answer you get when you multiply 18 by 32.

9. Take 17 times 4 away from 156.

14. Multiply 4 by 7 added to 19.

3. Generalisation I

Worksheets No. 3 and 4 (see Appendix 3) now become Nos. 5 and 6, and worksheet No. 5 (Appendix 3) is omitted, with the material contained therein being included in the new worksheets No. 5 and 6.

4. Generalisation II

No amendments. Worksheet No. 6 (see Appendix 3) becomes worksheet No. 7.

5. Notation

Amendment: The inclusion of further practice examples. Worksheets No. 7, 8 and 9 (see Appendix 3) become Nos. 8, 9 and 10.

Worksheet No. 11

Write down the print-out answers, in as many different ways as you can:

1. Add 5 to n.

2. Multiply x by 7.

6. Multiply a + 4 by 3.

7. Add a six times.

6. Consolidation

No amendments. Worksheet No. 10 (see Appendix 3) becomes No. 12.

Note

a. In some instances the completion of worksheets was set as a homework exercise.

b. In the class teaching (researcher) phase of the study (described in Chapter 5), the first and third year classes followed a less extensive programme (see text). Much of the work on notation was omitted with these classes, and Worksheets No. 8, 9 and 10 (i.e. Worksheets No. 7, 8 and 9 in Appendix 3) were consequently excluded.

APPENDIX 5

The Algebra Teaching Module (Final Version)

The module material is arranged as follows:

1. Rationale and outline of the teaching programme
2. Teacher's notes:
    a) Introduction
    b) Number Operations I
    c) Generalisation I
    d) Notation I
    e) Additional Components
3. Worksheets

The module differs from the earlier versions described in Chapter 4 and the class teaching (researcher) phase of the investigation described in Chapter 5 in the following ways:

1. The work previously organised under the headings 'Generalisation I' and 'Generalisation II' is now subsumed under 'Generalisation I'.
2. The work previously described under the heading 'Consolidation' now appears under 'Notation I'.
3. There is some re-arrangement of worksheets, and some condensation of the work on notation into a smaller number of worksheets.
4. The worksheets on the use of brackets described in Appendix 4 have been omitted (also see text, Chapter 5).

SESM Algebra: Teaching Module

Rationale

This teaching module has been developed on the basis of the findings of two research projects which set out to investigate the difficulties children were having in beginning algebra, and the reasons for these difficulties. These investigations revealed four main problem-areas in this topic, and the teaching module consequently aims to address each of these, as it was felt that ignoring any one could only lead to an incomplete understanding in the other spheres. The areas of difficulty are as follows:

1. Interpretation of letters. Children often do not understand that letters are representing numbers, and that the number represented may be a unique value (as in $x+2=5$), or an infinite range of values (as in $x+y=y+x$). They tend to cope with this problem by ignoring the letter completely, by substituting a particular value for it (perhaps based on an alphabetic code $a=1$, $b=2$, etc.), or by treating the letters as objects which can be merely collected up.

The teaching module seeks to address this problem by introducing letters at the generalised number/variable level, by which a given letter can take on a range of values (whole number and non-integer), and by which examples such as $x+2=5$ are treated as a special case in that here there is only one value which will make the statement true (though an infinite range of values which might in fact replace the letter). It was considered that children may find algebra easier if they were to start from the general case, rather than move from particular to general as is the more usual situation.

2. Symbolisation of method. Children often do not symbolise the methods used to solve problems in arithmetic. Consequently they are unable to produce a generalised form of that method, as is often required in algebra. The reason why children do not symbolise their methods appears to have three main components:

(a) Children often use their own 'common-sense' methods and not those taught in the mathematics classroom. These methods often do not lend themselves readily to mathematical representation.

(b) Even where the child uses a formal (taught) method, he/she may not be able to symbolise it appropriately.

(c) Even where the child can symbolise a formal method, he/she may not see that this is an appropriate thing to do.

Children are often more concerned with getting an answer than with paying explicit attention to the method. This may not matter in arithmetic, but in algebra, being able to represent the structure of a problem, operation or method is of central importance. If children do not have that structure available in the arithmetic case, they are unlikely to produce (or understand) it in the algebraic case.

The teaching module attempts to address this problem by concentrating on the method rather than the 'answer', and on the representation of that method. This is achieved by setting the teaching programme within the context of a 'mathematics machine' (cf. computer), whereby the aim of the programme is to write instructions to enable given problems to be solved by the machine. This use of a 'mathematics machine' also permits the introduction of letters as generalised number/variable by using letters as call-signs for number locations.

3. Legitimacy of the 'unclosed' answer. Children do not regard an expression such as 'n+3' as an 'answer', but rather as a sum which still needs doing. This can have two consequences:

(a) Children are reluctant to give 'n+3' as an answer, since to them it isn't. Consequently they will either give no answer at all, or substitute some value for n in order to obtain a numerical answer.

(b) Alternatively they attempt to perform the addition, and produce n3 or 3n as the answer, leading to confusion when the same abbreviation is introduced for the product nx3.

The teaching module addresses this problem by specifically considering the kinds of answer which the 'maths machine' can produce in response to an instruction such as 'add 3 to any number' (or 'n+3') when no replacement value for n is provided. Initially all operations are written in full (n+3 and nx3), and only later is '3n' introduced as an abbreviation for 'nx3'. Because of the strength of children's tendency to confuse '3n' with 'n+3', it is considered that the nonequivalence of the two expressions must be repeatedly examined and stressed. The fact that the 'answer' is in some cases the same as the 'sum' or 'instruction' is also an observation which children need to be explicitly aware of.

4. Use of brackets. Children see no need for the use of brackets and consequently do not use them, thus leading to error in algebra when more than one operation is involved. Children's resistance to using brackets appears to be based upon the belief that:

(a) operations are to be performed in the order written

(b) the particular problem will dictate which operation should be done first if different to the recorded order

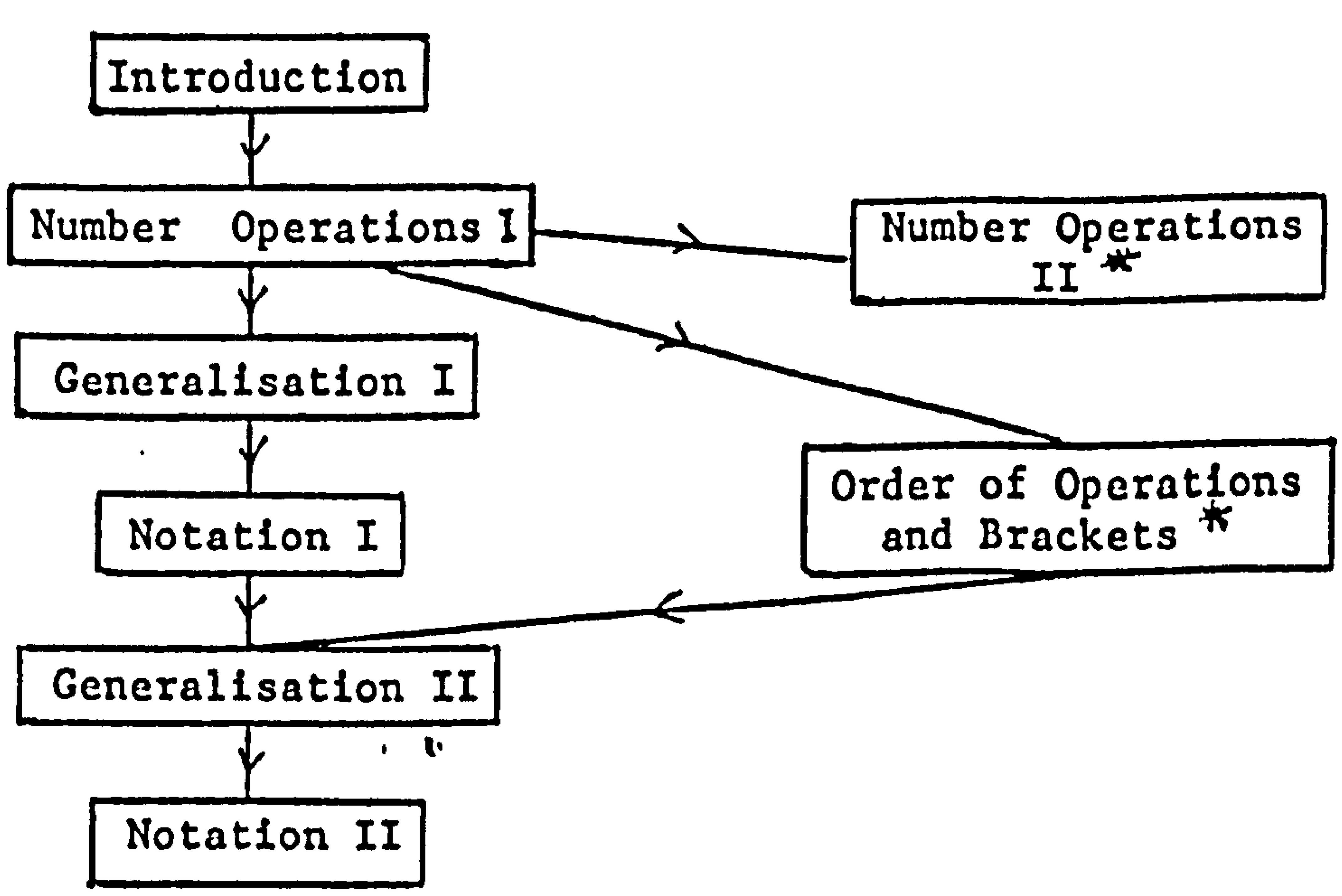
(c) the order is in any case irrelevant since the same answer will always be obtained.

The teaching module addresses this problem by incorporating a separate 'sub-unit' on the need to use brackets, and by stressing this need within the 'maths machine' context.

The above four areas of difficulty, though treated separately here for ease of discussion, are almost certainly interdependent. In particular, an important aspect which is reflected in each problem-area relates to the 'need' for or 'purpose' of using letters or giving general statements. Children often see no point at all in using letters, and in the case of simple equations (e.g. $x+3=5$) or substitution exercises ($a=3$, $b=5$, evaluate $a+b$), it may in fact be difficult to justify their value. It is considered that children will not take the use of letters seriously until the value of their use is made apparent to them. It is also considered that the use of the 'maths machine' context may go some way towards fulfilling this objective.

<u>The Teaching Module: Outline</u>

Components of the module are as follows. Those marked with an asterisk are supplementary to the main programme and contain optional extensions. These two sub-units may be done separately from the main programme, but the unit on 'Brackets' is best completed before 'Generalisation II'.



For the purposes of the SESM (Algebra) research, only the units 'Introduction' through to 'Notation I' are required. These units will therefore be described in full, and notes presented for the remainder.

Algebra Teaching Module: Introduction

Teacher's Notes

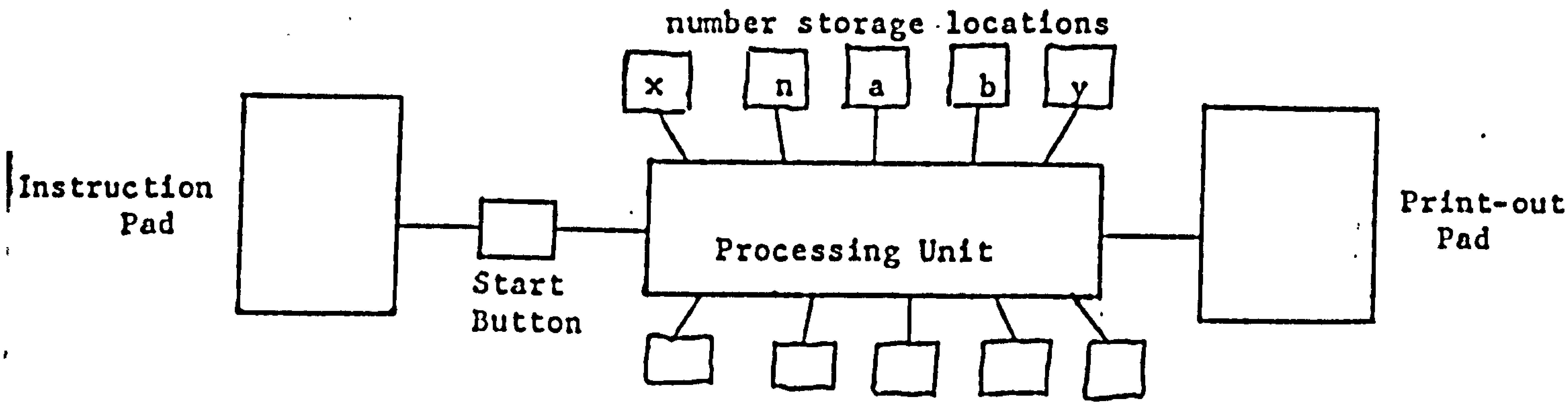
The programme is based upon introduction to a 'maths machine' (cf computer) which can be instructed to perform operations/solve problems. This machine is therefore a 'system' (cf algebraic system) with its own rules of operation which model those of algebra. The aim is to program the machine to perform given tasks/operations. All instructions must be given since the machine will only do exactly what the instructions say (it can't, for instance, read your mind to know what you meant:it will only do what you tell it to do). All instructions must be checked by 'reading back' as the machine would read them, before the start button is pressed. The principles upon which the programme is based are:

1. concentration on interpretation (letter as generalised number/variable; 'reading' of symbols and expressions)
2. concentration on method and the representation of that method
3. introduction to notion that an expression can be an 'answer' as well as a 'sum' or instruction, i.e. is another way of representing a value.
4. concentration on full recording of method, i.e. writing in all operations, over-use of brackets (brackets will always be used when there are mixed operations to indicate which operation is to be performed first. There will consequently be no need to specify any particular 'order of operations' convention).
5. concentration on equivalent forms of method (e.g. 3 x 4 and 4 x 3) and also on which forms are not equivalent (e.g. 12 – 5 and 5 – 12).
6. concentration on training on explicit 'algebraic' reasoning process.

The machine model (see diagram) consists of:

1. an instruction pad: the machine 'reads' the instruction on the pad and does what it says. The machine can only read mathematical expressions, so all instructions have to be given in maths language.
2. a processing component which does the actual working out (cf calculator)
3. a set of number storage locations labelled by letter call-signs, e.g. 'x' in an instruction calls up the number currently stored in that particular location.
4. a print-out pad where the machine writes out the results of carrying out your instructions (again in maths language which must be read and interpreted correctly).

MACHINE MODEL



The aims of machine programming are:

1. to write out instructions so that the machine will 'understand' them and perform the required task
2. to be able to read the print-out correctly and interpret what the machine has done.

Problem to be Addressed:

1. Introduction to 'Maths machine'
2. Use of brackets.

Materials Required

'Maths machine' model
Calculators for pupil-use
Worksheet No.1 + answer blanks

Teaching Sequence

1. Introduction to 'Maths Machine':

a) display model and explain that children are going to learn how to program the machine i.e. give it instructions so that it can work out various problems.

b) describe machine: it's similar to a computer and has the same main parts as does a computer. Describe the function of the -

(i) instruction pad (discuss various forms of giving machines instructions, e.g. punched cards, punched paper tape, magnetic tape like cassettes, etc): essential role of all these is to give the machine instructions on what to do.

(ii) start button

(iii) processing component (in what follows they will use the calculator to act as this part of the machine)

(iv) print-out pad (discuss different forms of print-out, e.g. teletype on T.V. screen; typewritten print-out sheets; magnetic tape, etc.)

(v) omit discussion of number storage locations at this stage.

c) discuss rules for programming machine:

(i) machine only does what instructions say

(ii) care must therefore be taken to make sure that instructions are written clearly and in full (importance of checking before processing)

(iii) all instructions must be in 'maths language'

(iv) if the machine is given more than one operation to do, it will not 'know' which one to do first. We therefore have to tell it which operation to do first by putting brackets round it. (In this unit brackets are introduced as something the machine needs. The need for brackets in general is left to the unit 'Order of Operations and Brackets'.)

2. Writing Instructions: Class Examples

a) Notes:

(i) concentrate on correct recording of operations, alternative ways of writing the same instruction, and which operations are not commutative

(ii) check recording of operation by 'reading back' before using calculator

(iii) check kind of answer expected (whole number, decimal number, 'large' value, etc.)

(iv) use calculator as 'processing unit' to calculate answer and record on 'print-out pad'.

(v) check kind of answer obtained: does it make sense, is that the kind that was expected, etc.

b) Demonstration Examples 1.

(i) I want the machine to add 15 and 8.

How will I write the instructions?

add 15 and 8 $15 + 8$

$8 + 15$

Discuss both expressions: are both correct, will I get the same answer, etc. Use calculator to check.

(ii) I want the machine to multiply 9 by 17

Multiply 9 by 17 9×17

17×9

Discuss both versions, check using calculators.

(iii) divide 24 by 8 $24 \div 8$ (read '24 divided by 8)

Discuss $8 \div 24$ — how expression is read, i.e. '8 divided by 24' meaning 'how many 24's in 8'
kind of answer expected (less than 1, i.e. decimal answer)
check with calculator

In this case expression cannot be written both ways. $8 \div 24$ means something different to $24 \div 8$ and will give a different answer.

Stress that sometimes I might want to divide 8 by 24: I don't always want to divide the big number by the small. If I want to divide 8 by 24 then I must write $8 \div 24$ and not $24 \div 8$, or I will get the wrong answer.

(iv) Take 7 away from 23 $23 - 7$ (read '23 minus/takeaway 7')

Discuss $7 - 23$ — how expression is read, i.e. '7 take away 23'
kind of answer expected (negative number)
check with calculator

Again in this case the expression cannot be written both ways, as $23 - 7$ means something different from $7 - 23$ and will give a different answer.

Sometimes I might want to take 23 away from 7 and get a negative answer, in which case I must write $7 - 23$, etc.

(v) I don't always have to use whole numbers, of course:
tell the machine to multiply 3.2 by 5.17:

multiply 3.2 by 5.17 3.2×5.17

5.17×3.2

c) Class Exercises 1 (do with class, recording instructions and answers - using calculators - on front side of answer blank):

(i) add 9 and 27

(ii) multiply 1.4 by 8

(iii) take 9 away from 28

(iv) divide 24 by 6

(v) divide 3 by 12

(vi) take 5.9 away from 1.7

In each case, ask
can expression be written another way, discuss meaning of alternative forms, kind of answer expected (decimal, negative etc.)
check using calculator

d) (optional): children to provide verbal instructions, other children to supply mathematical equivalent, e.g. in 'team game' setting etc.

e) Demonstration Examples 2: more than one operation. The rule introduced here will be that brackets are always needed to tell the machine which operations to do first.

(i) add 15 and 9, and then add 7:	(15+9) + 7*
	7 + (15+9)
(ii) multiply 2.1 by 4 and then add 5.25:	(2.1x4)+5.25
	5.25+(2.1x4)
(iii) divide 19 by 4.7 and then take away 10:	(19 ÷ 4.7) − 10
(iv) add 3.4 and 2.81 and take your answer away from 19:	19 − (3.4+2.81)
(v) add 15.5 and 17.3 and double the answer:	(15.5 + 17.3) x 2
	2 x (15.5 + 17.3)

*In the case of all 'adds' and all 'multiply's' e.g. 15+9+7, 2x3x5 etc., the same answer will be obtained no matter which part is done first (check). In this case, therefore, you can leave out the brackets. However, it is not wrong to put them in, so if in doubt, always use brackets. In every other case (e.g. 12-7-4, 100 ÷ 10 ÷ 5, 2x3+5 etc.) you will not get the same answer so that brackets must be used to say which operation is to be done first. In each question check expression by 'reading back' to check with original instructions. Using calculators, record the print-out answers. Would we have got the same answer if we had done the other operation first? Use calculator to show that a different answer would be obtained: this is why the brackets must be put in, to show which answer we want to get.

3. Writing Instructions: Worksheet No.1

Children to work independently, giving both forms of expression where appropriate. Children to complete instructions first, checking by 'reading back' to themselves first (what do instructions mean), before using calculator to record print-out answer. Instructions and answers to be recorded on reverse side of answer blank.

Check alternative versions; discuss meaning of alternatives, whether both versions are possible, reason why not if not equivalent, etc.

Algebra Teaching Module: Number Operations I

Problem to be Addressed

1. Availability of formal method
2. Symbolizing formal method.

Principles of Teaching Programme

1. Required operation is decided by the structure of the problem and not, for example, by saying 'divide the big number by the small one'. This point is addressed more fully in 'Number Operations II'.

2. The commuted form of an expression is always to be recorded in addition to the original form where valid. Children are not always clear which expressions are commutative and which are not. By recording both versions where appropriate, and by discussing cases where this is not valid (including 'proof' by calculation of the answers), it is hoped that children will be made aware of the difference, and also of the need for correct form of recording.

Materials Required

'Maths machine' model
Calculators for pupil use
Worksheet No. 2 + answer blanks

Teaching Sequence

1. Solving Problems: Class Examples

a) Demonstration Examples

Discuss how instructions should be recorded for each example. As before, discuss alternative forms and which ones are valid:

(i) I've got 27 records, my sister has 34. How many records do we have altogether?

(ii) I collect 35p from everyone to buy a present. If I collect from 18 people, how much (in pence) will I have?

(iii) My brother has £123. That's £49 more than I've got. How much have I?

(iv) Petrol costs £1.54 a gallon. I buy 8 gallons. How much will it cost?

(v) I've bought 7 gallons of petrol and it cost £11.76. How much a gallon was it?

b) Class Examples

Use front of answer blank to record instructions. Both forms of expression are to be recorded where appropriate.

Children can use their calculators to fill in the 'print-out' section afterwards.

Children to provide word problems, class to record required instructions in each case. Try to encourage a variety of operations. There is room on the answer blank for 6 problems.

As before, check that both forms of instruction are given when both are valid, and discuss why the alternative form is not valid in subtraction, division examples (this requires consideration of the structure of the problem).

2. Solving Problems: Worksheet No.2

Working independently, children to record instructions first, using reverse side of answer blank. Encourage children to check their instructions by interpreting back before using the calculator to obtain print-outs. As before, children to record both forms where appropriate.

Check both forms of instruction are given where valid, discuss why alternative form is not valid in subtraction, division examples.

Algebra Teaching Module: Generalisation I

Problem to be Addressed

1. Interpretation of letters
2. Formalization of general method
3. Legitimacy of representing an 'answer' by an expression or 'procedure call (e.g. n+3)
4. Justification for use of letters and general statements.

Principles of Teaching Programme

1. Concentration on interpretation of letters as generalised number/variables, i.e. a given letter represents an infinite range of values.
2. The same letter in the same expression represents the same value.
3. Different letters can represent the same or different values, i.e. there is a choice (children find this idea difficult, partly because they tend to think in an 'either-or' fashion, i.e. either it's the same or it's different. The notion is thus introduced rather gently at this stage).
4. Children tend to think in terms of specific cases rather than generalities, e.g. 'any rectangle' is used by the teacher to mean the general case, i.e. what can be said of every rectangle in the world. Children, however, tend to interpret 'any rectangle' or 'any number' to mean 'any particular one you choose'. Consequently they will think of, and give as an answer, the particular example they thought of (e.g. a rectangle 5 by 8) and think that this is what was intended. It will be necessary, in discussing general statements, to discuss this difference in interpretation, e.g. if I write 5 x 8 for the area of any rectangle, this only tells me something about one particular rectangle. It doesn't give me a rule for every rectangle in the world. To do this, I need to write something like n x m or a x b, where n stands for how long the rectangle is and m for how wide it is.

   The aim in what follows will be to give the machine rules for answering every problem of a given kind that I give it.
5. All operations to be written in full.

Materials Required

Maths machine model

Worksheet No.3,4,5 + answer blanks

Calculators

Teaching Sequence

1. Introduction

a) Revision of 'maths machine' components, using model

b) Introduction to idea of number storage locations - places in the machine where numbers can be held. The numbers in these places are given labels or 'call signs', i.e. a letter. Using a letter in a machine instruction will 'call up' the number stored, e.g. if I give the instruction 'add x', the machine will add on whatever number is in the x-location.

2. Use of Letters/General Statements: Class Examples

a) Demonstration Examples

(i) I want the machine to add 5 to any number I give it, so I need to write a rule for the machine to do this

Can I write 6 + 5? No, this only tells the machine to add 5 to 6. The machine hasn't got a rule for adding 5 to any number in the world.

Instruction: x+5 (or 5+x) (or a+5 etc)
This means 'add 5 to whatever number x stands for' and x can be any number in the world.
The machine now knows it has to add 5 to whatever number x is.
If I now press the start-button, the machine will look to see what number is in the x-location and add 5 to it.
(Just discuss the 'instructions' at this stage. We'll come to the 'print-out' later).

(ii) I want the machine to multiply any number I give it by 7.

Instruction: n x 7 (or 7 x n) etc.

Discuss meaning: multiply whatever number n is by 7.
Again, discuss instructions such as '3 x 7': and the difference between this and n x 7.

(iii) Take 8 away from any number I give

(iv) Divide any number I give by 13

(v) Take any number I give away from 47

(vi) Divide 2000 by any number I give and then add 50 (remind re use of brackets).

Continue to reinforce distinction between commutative and non-commutative expressions.

b) Class Exercise

Using front of answer blank to record instructions (6 examples), children to supply examples of above type and class to record instructions using letters to represent general number values. Continue to record both forms of commutative expressions. Record all operations in full (e.g. 3 x n not 3n).

3. <u>Use of letters: Worksheet No.3</u>

Working independently, children to record instructions only, giving both forms of expression where appropriate.

Discuss instructions given, meaning of letters, examples of values letters might stand for (are there any numbers besides whole numbers, etc.), meaning of brackets. Why are instructions y+8, for example, and not e.g. 13+8? What does y+8 mean?

4. <u>Legitimacy of Expression as Answer</u>

a) Introduce idea of replacement values. Using instructions recorded by children on front of answer blank in 2(b) above, discuss what the print-out will be for different values assigned to the letters. Values are assigned by writing e.g. x=5 underneath the general instruction on the instruction pad:

e.g. Instruction: x x 4		Print-out:	
x = 5 ⟶ Start	⟶	20	) Write answer
x = 2.3 ⟶ Start	⟶	9.2	) on a level with
x = 100 ⟶ Start	⟶	400	) assigned value.
etc.			

(Each new value replaces the old one)

b) Repeat for each of the 6 examples on the front of the answer blank used in 2(b) above. Take 2 or 3 replacement values suggested by the children in each case, recording the replacement value as 'n=7.5' etc. on the instruction pad, and writing in the appropriate print-out in each case (using calculators where necessary).

c) Introduce idea of 'general answer': if <u>no</u> value is given for the letters, then the machine will not print out a particular number answer (why?). The machine will therefore print out the 'general answer' e.g. x x 4, meaning 'whatever number x is, multiplied by 4' or 'the number which is 4 times x'.

e.g.

Instruction: x x 4 ⟶ Start ⟶ Printout: x x 4

Repeat for each of the 6 examples used above (4b). Answer blank should now read:

Instruction Pad		Print-out Pad
x x 4		x x 4
x = 5	— — Start — —	20
x = 2.3		9.2
etc.		

d) Using the instructions recorded for worksheet No.3, children to complete the print-out section as follows:

(i) What will the machine print out if no value is given for the letters? (Why won't it give a particular number answer?)

(ii) Select 2 replacement values for the letters in each case, recording these on the instruction pad (or allow children to choose their own values), and write in the corresponding print-out answers.

5. Use of More Than One Number Location: Worksheet No.4

a) Children to attempt worksheet on their own first, writing in the instructions only. Discuss Nos. 1 (e.g. a+a) and 7 (e.g. x+y) if necessary. Will brackets be needed?

Discuss instructions: When does question require repeated use of the same letter and when are different letters needed. What is the difference between (for example) a+a and x+y? Introduce idea that different letters do not necessarily mean the values must be different: they can be different or they can be the same, e.g. there is nothing to stop me putting the number 3 in both x and y locations. The difference is that in a+a the number must be the same; in x+y I can have either the same number or different numbers (i.e. I have the choice). Check use of brackets.

b) Children to complete print-out section for Worksheet No.4, recording:

(i) the answer the machine will give if no value is given for the letters

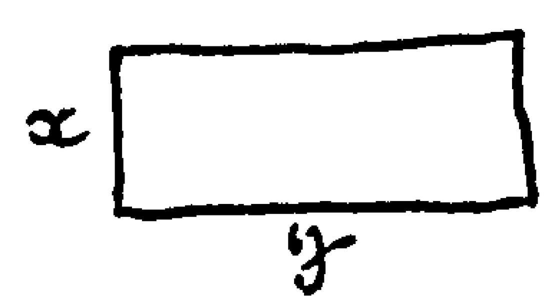
(ii) the answers given in each example for one or two teacher-given or pupil-chosen replacement values for the letters used. (Remind children that these values do not have to be whole numbers).

6. Problem-solving: Worksheet No.5

(a) Discuss 3 or 4 examples with the class.

Record all operations in full:

e.g. 1) Write instructions for finding the area of any-sized rectangle.

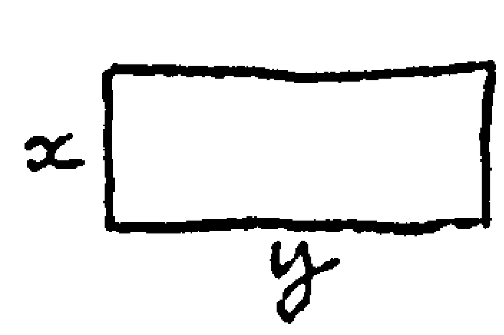


What is area?
How do you find the area of a rectangle,
What do you have to do?
How shall we give the instruction 'multiply length by breadth' to the maths machine?

Why not write e.g. 10 x 15 as the instruction?

Record instructions only at this stage.

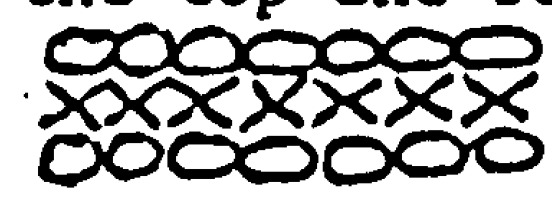
2) Write instructions for finding the perimeter of any rectangle.



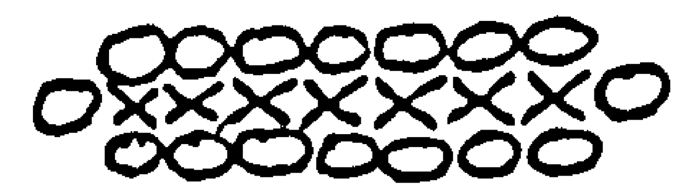
What is perimeter?
How do we find the perimeter of a rectangle?
How shall we give the instructions 'add all the sides' to the machine?
Why not write e.g. 6+4+6+4 as the instruction?

3) A factory embroidery machine does a design in cross-stitches and circles.
First it stitches a row of crosses: 

Then it stitches a row of circles along the top and bottom:



and then circles at each end:



Write instructions to find how many circle stitchers I will need for any number of crosses I want to use.

If I use 17 crosses, how many circles will I need?

The aim in this example is to find the rule by 'reasoning', e.g. 'it must be twice the number of crosses plus two'. (Remember brackets). However, introduce the strategy of finding the rule by looking at several numerical examples ('another way of doing it is .....'). Point out that sometimes this is an easier way of doing it.

e.g.:

No. of crosses		no. of circles
3	$\longrightarrow$	8
4	$\longrightarrow$	10
5	$\longrightarrow$	12
10	$\longrightarrow$	22 etc.

therefore rule must be 'double and add 2' i.e. $(2 \times n) + 2$

Use numerical example to check that the general expression recorded is correct, i.e. does work.

N.B. Record numerical cases as examples of the general expression:

Instructions: $(2 \times n) + 2$ Print-out: $(2 \times n) + 2$
$n = 17$ 36

At each stage stress the following steps in problem-solving:

a) what do I want the machine to find?
b) what instructions are needed (how do we do it?)
   - do I need any number locations/how many?
   - what operation(s) needed?
c) how do I write the instructions/method?
   - do I need brackets?
d) what have I written/what does it mean?
   - is that what I want?
   - is it correctly recorded?
e) what will the print-out (answer) be?
f) what does the print-out mean?
   - is it correct?
   - check by number substitution

Children to complete Worksheet 5, recording the instructions only on the reverse side of the answer blank.

Discuss instructions. Check both forms for commutative expressions.

Discuss use of letters, e.g. question 3 - find perimeter of any triangle Why not write e.g. a+a+a? Does a+b+c cover the case of when all the sides are the same length? (N.B. if you wanted the perimeter of a triangle where all the sides were definitely different, you would have to write e.g. a+b+c, $a \neq b \neq c$).

Discuss what print-out would be (in most cases the 'answer' is the same as the instructions in this particular set).
Some of these answers can be written in a shorter form. Lead into Notation I.

Algebra Teaching Module: Notation I

Problem to be Addressed

1. Equivalence of a + a + a + a and 4 x a
2. Representation of 4 x a by 4a
3. Non-equivalence of 4 + a and 4a

Principles of Teaching Programme

1. Reinforce the notion of letters as representing numbers by frequently referring to 'a' as 'the number which a stands for' and by occasionally asking children to provide examples of what 'a' is (not necessarily whole numbers).
2. The replacement of a + a + a + a by 4 x a is justified on the grounds of being shorter and quicker to write.
3. The replacement of 4 x a by 4a is really arbitrary, but can be given some kind of justification on the grounds that you could confuse the two answers '4 x a' and '4 + a', whereas '4a' and '4 + a' are easier to distinguish.
4. Children do not readily recognise the equivalence of 4 x a and a + a + a + a, especially in reverse. This should be assisted by frequently translating 4 x a as both '4 times the number which a stands for' and '4 lots of the 'a' number' which is 'a + a + a + a'.
5. Children have a strong tendency to regard '4a' as a joining together of 4 and a, i.e.as 4 + a rather than 4 x a. Hopefully this programme should discourage this. However, it is suggested that initially (and at intervals in later work) the presentation of the term 4a should be accompanied by a verbal interpretation '4 times the 'a' number'. In addition, for answers such as 4 + a ask, 'can this be written as 4a? Why not?' etc.
6. Introduce idea of checking expressions by substituting numbers (especially important when brackets are involved).

Materials Required

Maths machine model (optional)

Worksheets No. 5,6,7,8,9.

Teaching Sequence

1. Class Examples

a) Demonstration Examples

Refer to Worksheet No.5; the 3 demonstration examples on the front of the answer blank:

1) area of rectangle

Instruction		Print-Out
x x y		x x y
	OR	xy

2) perimeter of rectangle

Instruction		Print-Out
x + y + x + y		x + y + x + y
	OR	2 x x + 2 x y
	OR	2x + 2y

3) *stitches*

Instruction		Print-Out
(2 x n) + 2		(2 x n) + 2
	OR	2n + 2
		4n? (why not?)

Further examples:

4) find the perimeter of a 5-sided shape with all sides the same length.

	Instruction		Print-Out
	a + a + a + a + a		a + a + a + a + a
OR	5 x a	OR	5 x a
		OR	5a

5) What would the instructions be if the 5 sides could be <u>different</u> lengths? What would the print-out be?

<u>NB</u> the 'answer' is usually written in the shortest form, i.e. 5a rather than 5 x a or a + a + a + a + a.

b) <u>Class Examples</u>

Using reverse side of answer blank and the instructions recorded for Worksheet 5, discuss the 'print-outs' for each question, writing in all possible versions and stressing that the 'answer' is usually given in the shortest form.

Check that sums such as a + b cannot be written 'ab' (why?)

2. <u>Worksheet No.6</u>

Children to record the shortest print-out answer for each instruction. Discuss answers, checking the interpretation of '5n' etc. (5 x n and n + n + n + n + n) and checking why n + m cannot be written 'nm'.

3. <u>Worksheet No.7</u>

Using the reverse side of Worksheet No.6, children to record the instructions for each question, and the print-out answers, giving the shortest form.

In discussing the instructions, discuss all alternatives, e.g. a + a + a and 3 x a or a x 3, remembering to check with children what the letters used represent, asking for examples etc.

Discuss answers, again checking interpretation of 3n etc. and fact that n + a cannot be written 'na'.

In question 11, check answer 3n + 2a. Could that be written any shorter? Consider various alternatives, and what is wrong with each: e.g. 5na '5 times n times a' - do we have 5 lots of the 'n' number in the shape, etc. Check by substituting numbers. Also check Sheet 6, Nos. 3, 7.

4. <u>Worksheet No.8</u>

a) Children to record instructions (using 'maths') on instruction pad side, then record print-out answer, giving answers in the shortest form. Check use of brackets, interpretation of expressions, possible alternative forms etc. Check any doubtful answers by both discussing what the answer means, and by substituting numbers.

b) Optional: using reverse side of sheet No.8, children to make up instructions like those on worksheet 8, giving them to their neighbour to fill in the 'maths' instructions and answers.

5. <u>Worksheet No.9</u>

a) Some of the answers given on the print-out side are wrong. Children to cross out all incorrect answers.

Where possible, children to write in the 'shortest form' answers.

Again discuss meaning of correct answers, and why incorrect answers are wrong.

b) Optional: Using reverse side, children to make up instructions together with correct and incorrect answers as on Worksheet 9, giving them to their neighbour to mark the correct and incorrect answers.

6. <u>Worksheet No.10</u>

This is to provide practice in handling symbolic expressions and their simplification. Children are to record the 'shortest form' print-out answers. In discussing the answers, discuss first the meaning of each instruction, what the letters stand for, whether only one value is involved (e.g. 4a+5a) or whether two possibly different values are involved (e.g. 3a+2b) and why the former can be simplified to 9a (4 lots of the 'a' number plus 5 lots of the same number, also check by substitution) but the latter can't be simplified (5ab, for example, would mean '5 lots of the 'a' number and we only have 3 lots in the expression, also check by substitution).

SESM Algebra: Additional Module Components

Though not required by the demands of the project's aims, which were to suggest a teaching programme which would only address certain specified errors, the following module components are suggested as providing a more rounded programme which will both extend and reinforce the understandings hopefully developed in the 'main core' programme outlined in the above 'Introduction' through to 'Notation I'.

In order to ensure that children do make firm and thereby retain their understandings so far attained, and in order to enable those who assimilate the ideas involved more slowly to complete the process of assimilation, it is considered that all the following module components with the possible exception of 'Number Operations II' which is less essential to the development of specifically algebraic notions at this stage, should ideally be covered. However, as they are not strictly essential to the requirements of the project as such, the details and examples appropriate to each component are excluded at this stage, and the following outline only is given:

1. Number Operations II

This will go into more detail concerning the recording of number operations and the interpretation of word problems in order to select the number operations appropriate to their solution.

In recording number operations children have difficulty in particular with subtraction and division statements. This difficulty appears to be based mainly on three problem areas which need to be addressed by the teaching programme:

i) some children interpret $6 \div 72$ as '6 divided into 72', perhaps confounding the expression with the $6\overline{)72}$ form of recording division. '4 - 17' is similarly translated as 'take 4 away from 17'.

ii) many children adhere to the rule that 'you always divide the large number by the small one (or take the small one from the large number)'. Consequently they will either always write the operation in that form, regardless of the requirements of the question, or they consider that the order in which the operation is written doesn't matter, as it will always be performed in accordance with the above rule.

iii) some children believe that all operations are commutative. Consequently they regard such expressions as $391 \div 17$ and $17 \div 391$ or $6.44 - 8.37$ and $8.37 - 6.44$ as equivalent, and will hence write either.

Selecting the appropriate number operation requires consideration of the structure of the problem, and of what the various operations actually mean. It is considered that one way of addressing this problem is to look at several different examples of addition, subtraction, multiplication and division problems to see what operation is involved and why; also to see what various multiplication etc. problems have in common. Children can also be given 'classification' exercises in which they select out all the addition etc. problems from a given set.

2. Order of Operations and Use of Brackets

The aim of this module component is to generalise the idea of the need to use brackets so that children do not perceive it only as a requirement of the 'Maths Machine' or computer context. The component will therefore be based on the consideration of the answers that will be obtained if given expressions are evaluated in different orders. The recognition that different answers will be obtained (except for all 'adds' and all 'multiply's') is considered a necessary prerequisite to acceptance of the need to use brackets.

Children will also be given exercises in which a particular value is given and the brackets have to be inserted in the corresponding expression in accordance with the value given. This activity can also be conducted via card games such as 'Krypto' (initially using a simplified version).

3. Generalisation II

This provides more investigation activities in which the aim is to derive the rule or instructions necessary in order to solve every particular numerical example of that problem type.

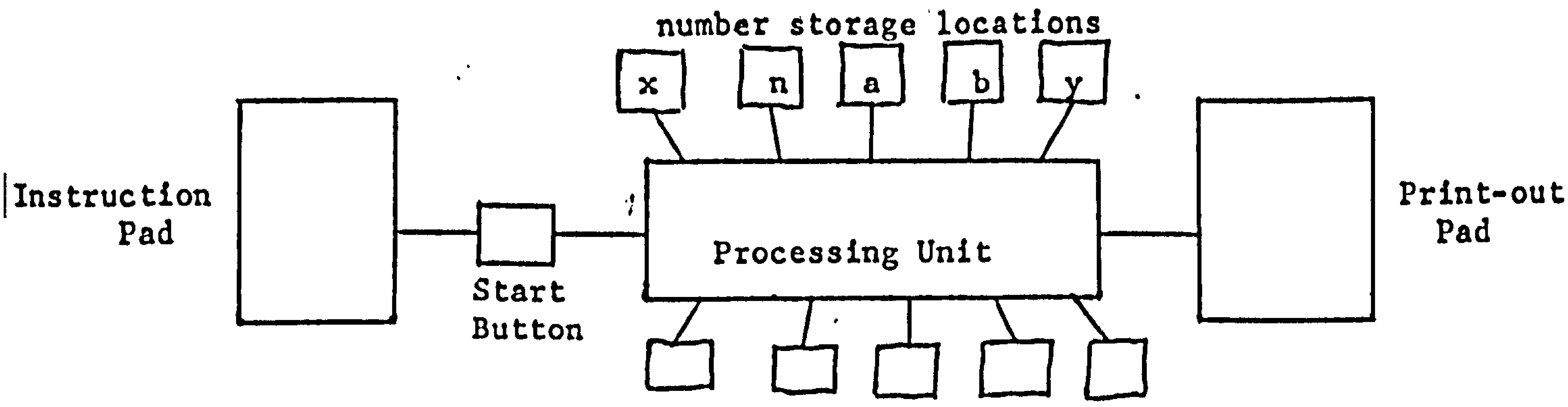
The exercises thus reinforce all the notions previously introduced concerning use of letters, and use of brackets and appropriate simplifying notation.

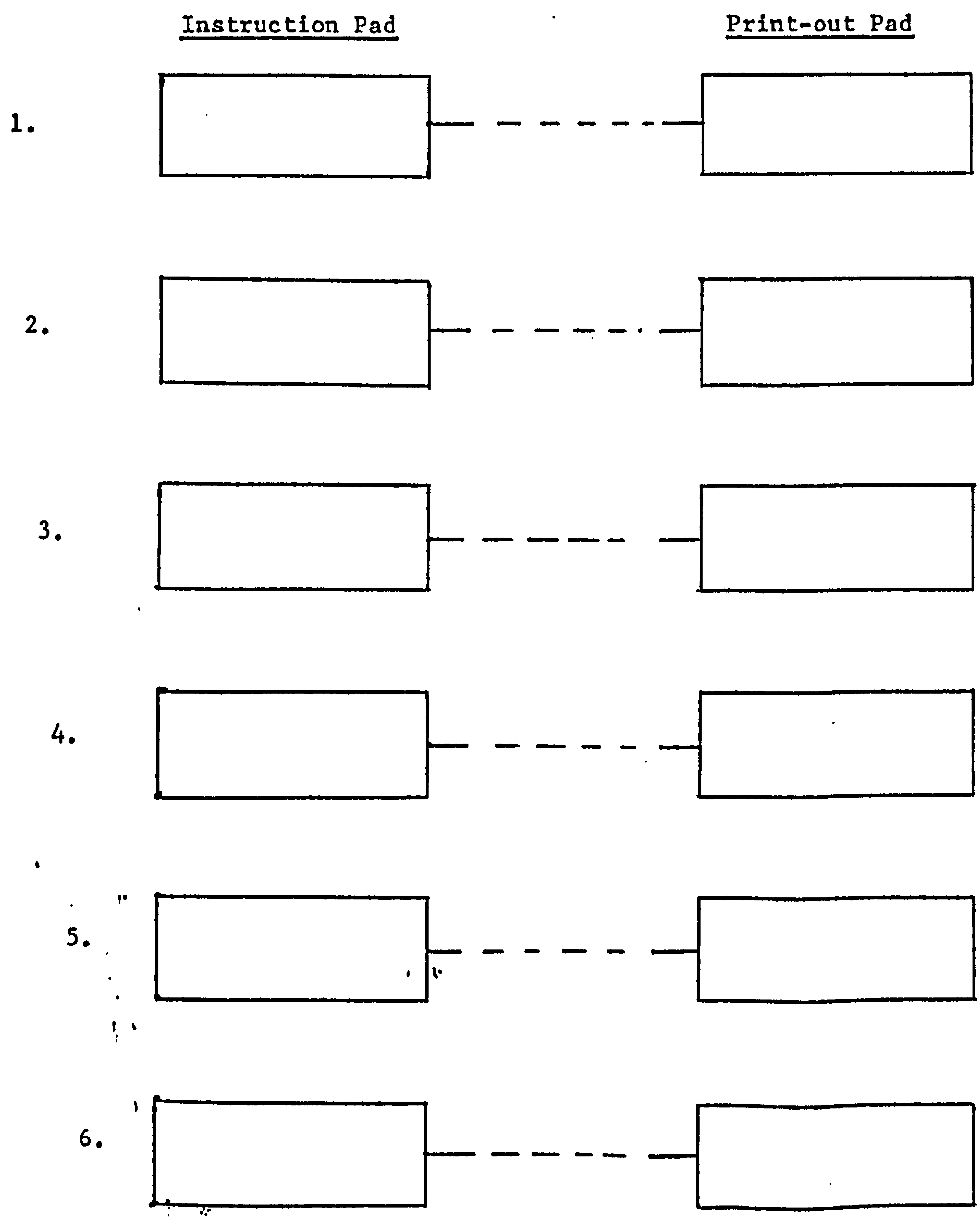
4. Notation II

Part I will provide more practice on simplifying answers. Item types involved will include such expressions as 3+n+4, (3xn)+4, 3x(n+4), 3xnx4, (3xn) + (4xn), and (3xn) + (4xm).

Part 2 will introduce the notion of replacing eg. a x a x a by a^3, and the distinction between the meanings of a^3, 3a and a+3.

MACHINE MODEL





MACHINE MATHS - SHEET 1

Write down the instructions you would have to give the machine if you wanted it to do the following problems. Write the instructions both ways if possible. CHECK that you have written what you mean.

1. Add 19 and 38
2. Take 16 away from 33
3. Multiply 8 by 51
4. Divide 24 by 3
5. Add 9 and 28 and then take away 15
6. Add 2.7 and 5.82
7. Add 9.8 to the answer you get when you multiply 3.65 by 7.8
8. Take 19 away from 57 and then multiply the answer by 4
9. .Take 497 away from 153
10. Divide 10 by 4. What kind of answer will you get?
11. Divide 4 by 10. What kind of answer will you get?
12. Multiply 7 by 91 and then take away 18
13. Subtract 15.3 from 49.7. What kind of answer will you get?
14. Subtract 49.7 from 15.3. What kind of answer will you get?
15. Divide 2.4 by 9.7

Now use your calculator to work out each answer and write your answers in the 'print-out pad' box.

MACHINE MATHS - SHEET 2

1. I drive 67 kilometres before lunch, and 98 kilometres after. How many kilometres do I drive altogether?

2. I want to save £185. So far I've saved £127. How much more must I save?

3. I have 33 pupils in my class and I have to give them each 14p for their bus fare. How many pence do I need?

4. A rocket ship goes 93 miles in 1 second. How far will it go in 17.5 seconds?

5. Spurs scores 194 goals in one season. Preston scores 238. How many goals do the two teams score altogether?

6. I've got 372 records which I decide to give to 6 friends so that they all get the same number. How many will each person get?

7. 240 people are coming to see a show. The chairs are to be in rows, and each row has 15 chairs. How many rows will be needed?

8. I've got a large bar of chocolate which has 136 squares of chocolate in it. There are 17 squares of chocolate in a row. How many rows are there?

9. A cup holds 4.8 ounces of flour. I empty 18 cups of flour into a bowl. How many ounces have I put in the bowl?

10. A crate of soft drink holds 72 cans. I order 52 crates. How many cans of soft drink will I get?

11. A fat man wants to lose 57 lbs. weight. So far he's lost 29.5 lbs. How many more lbs. must he lose?

12. I've got £42.60. My brother has got £17.90 more than me. How much has he got?

13. A train travels 1792 miles. I know it travels 58 miles every hour. How many hours did it take?

14. The length of a rectangle is 15.4 cms and its width is 7.6 cms. What is the area of the rectangle?

15. Make up a problem, and then write down the machine instructions for it.

MACHINE MATHS - SHEET 3

1. Add 8 to any number I give.
2. Take 17 away from any number I give.
3. Multiply any number I give by 13.
4. Multiply 16 by any number I give.
5. Divide any number I give by 7.
6. Add 18 to any number I give, and then multiply the answer by 4.
7. Subtract 9 from any number I give, and then multiply by 15.
8. Double any number I give.
9. Divide 329 by any number I give.
10. Divide any number I give by 48.
11. Divide any number I give by 18, and then add 22.
12. Take any number I give away from 100.
13. Divide 99 by any number I give, and then subtract 16.
14. Multiply 61 by any number I give, and then take the answer away from 999.
15. Add any number I give to 23 and then double the answer.

MACHINE MATHS - SHEET 4

1. Add any number I give to itself.
2. Multiply any number I give by itself.
3. Add any number I give to itself, and then multiply by 3.
4. Divide any number I give by itself.
5. Multiply any number I give by itself, and then subtract 15.
6. Think of a number, multiply it by 7, and then take away the number you first thought of.
7. Add together any two numbers I give.
8. Multiply together any two numbers I give.
9. Divide any number I give by any other number I give.
10. Multiply any two numbers I give, and then add 17.
11. Multiply any two numbers I give, and then add another number.
12. Add together any three numbers I give.
13. Add any number I give to itself, and then multiply by another number.
14. Multiply together any three numbers I give.
15. Multiply any number I give by itself, and then take away another number.

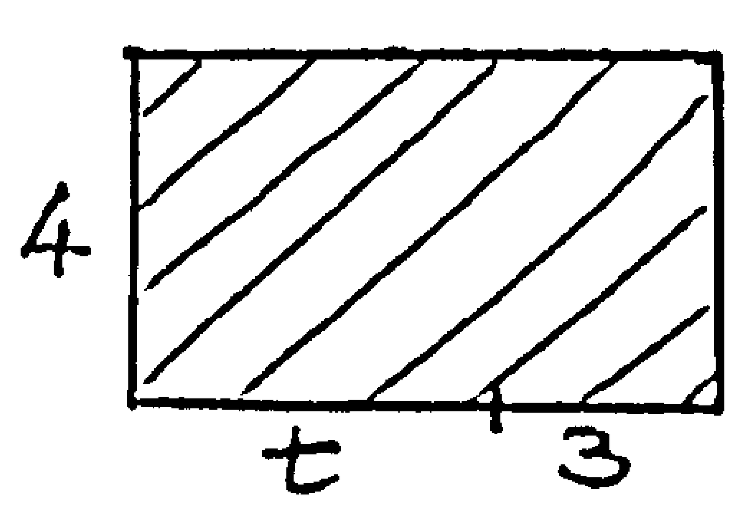
MACHINE MATHS - SHEET 5

Write instructions for the following:

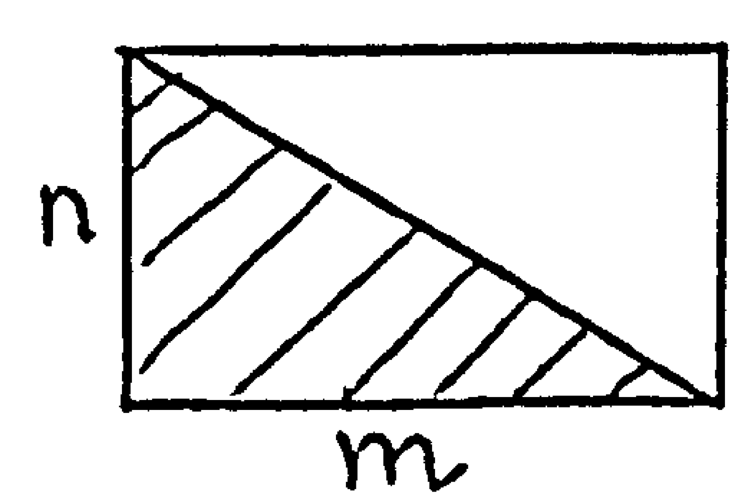
1. Find the area of any rectangle
2. Find the area of any square
3. Find the perimeter of any triangle
4. Find the perimeter of a triangle with 2 sides the same length
5. Find the perimeter of a triangle which has all its sides equal
6. Find the perimeter of a square
7. A rocket goes p miles in one second. How far will it go in t seconds.
8. A bus fare is x pence for each stage. How much does it cost to go y stages.
9. Find the perimeter of a shape which has n sides which are all 3cm long
10. Find the average of three numbers

Write instructions for finding the area of the shaded shapes:

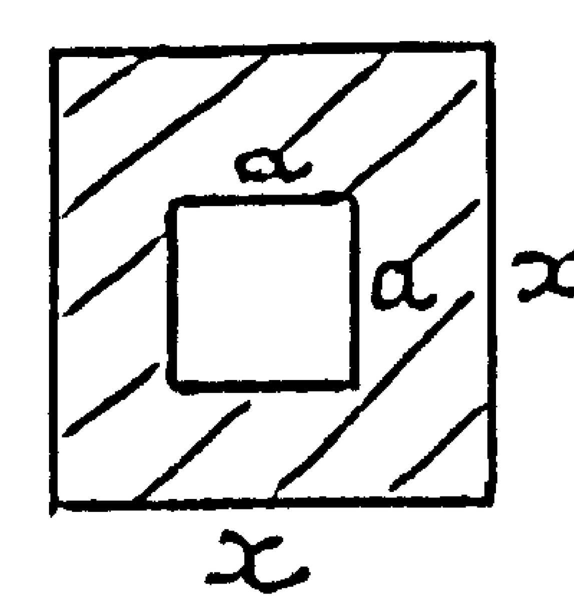
11.



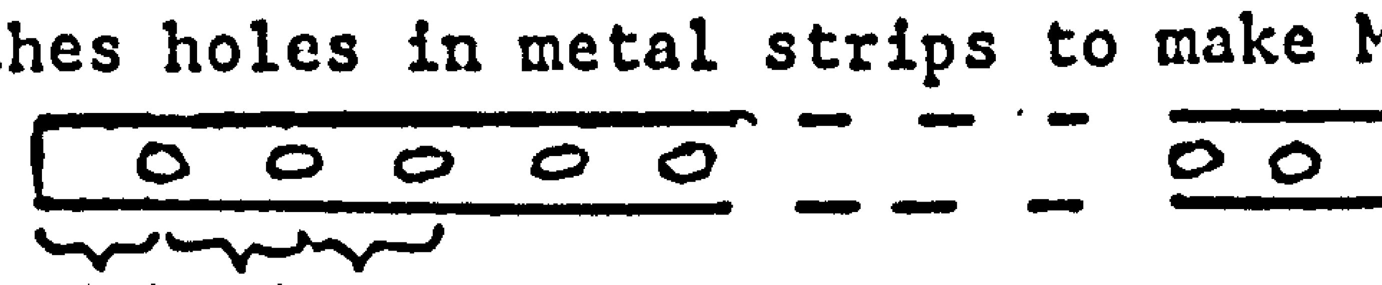
12.



13.



14. A factory machine punches holes in metal strips to make Meccano sets:

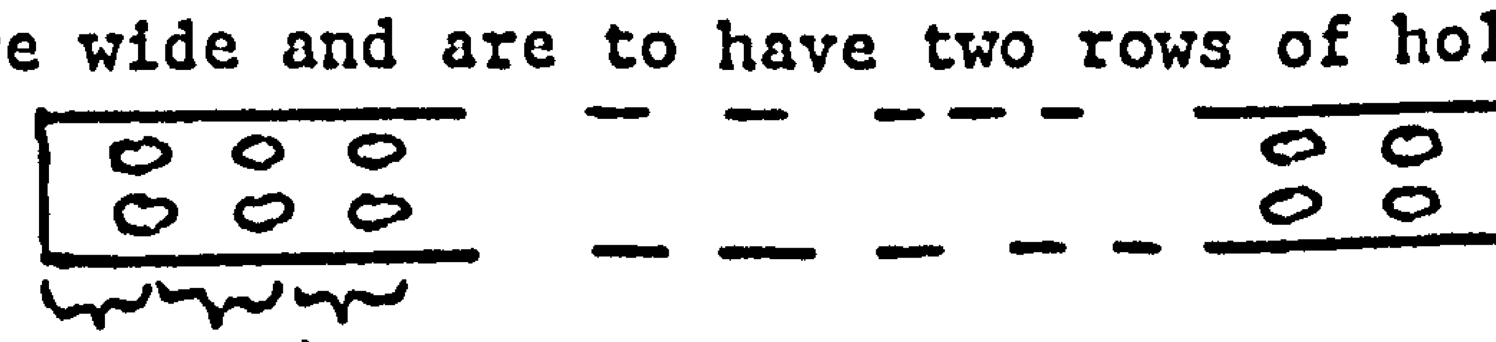


The holes are to be all 1 inch apart and start 1 inch from the ends.

Write instructions to find how many hole punchers I need to set for any number of inches length of metal strip.

How many hole punchers will I need for a strip which is 27 inches long?

15. Some strips are wide and are to have two rows of holes:



Write instructions to find how many hole punchers I will need for any number of inches length of metal strip this time.

How many hole punchers will be needed for a strip which is 15 inches long?

MACHINE MATHS – SHEET 6

	Instruction Pad	Print-out Pad
1.	$a+a+a+a$	
2.	$3 \times y$	
3.	$n+a+n$	
4.	$m \times n$	
5.	$x \times y \times 4$	
6.	$p+p+p+p+p+p$	
7.	$a+a+b+b+b$	
8.	$(3 \times n)+p$	
9.	$3 \times (n+p)$	

MACHINE MATHS - SHEET 7.

Find the perimeter of each shape:

1. 4-sided shape, all the sides must be the same length.

2. 6-sided shape, all the sides can be different lengths.

3. 6-sided shape, all the sides must be the same length.

4. 3-sided shape, all the sides can be different lengths.

5. 3-sided shape, all the sides must be the same length.

6. 5-sided shape, all the sides can be different lengths.

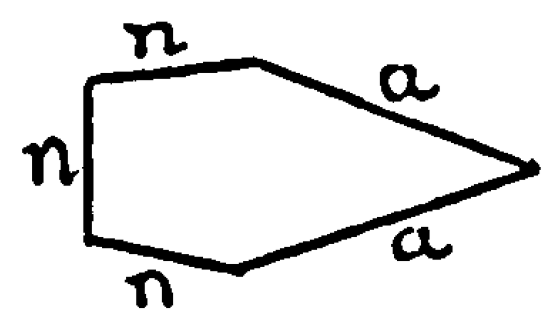
7. 10-sided shape, all the sides must be the same length.

8. 24-sided shape, all the sides must be the same length.

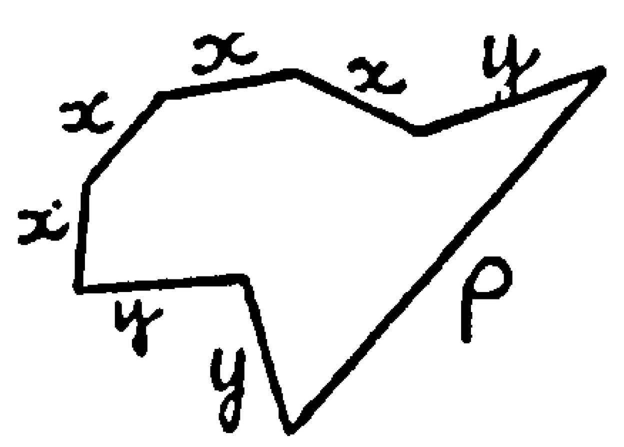
9. A shape which has 14 sides altogether, 8 of which are all one length, and 6 sides are all another length.

10. A shape which has 20 sides altogether, 13 of which are all one length, and 7 sides are all another length.

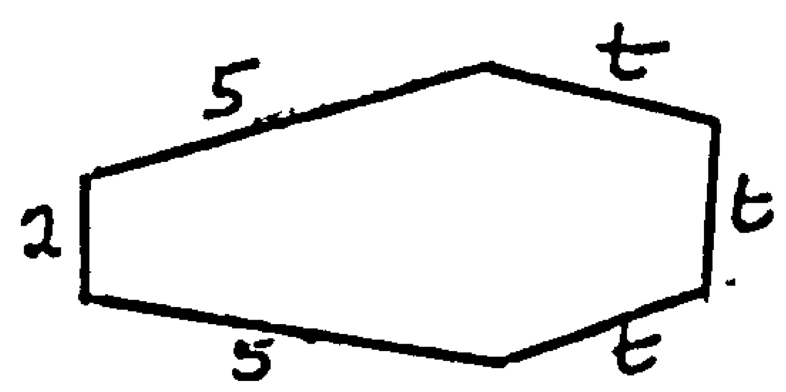
11. Shape:



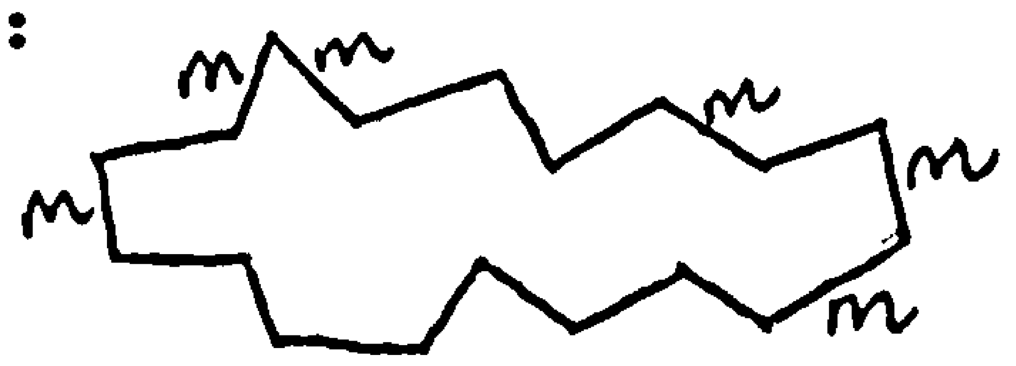
12. Shape:



13. Shape:

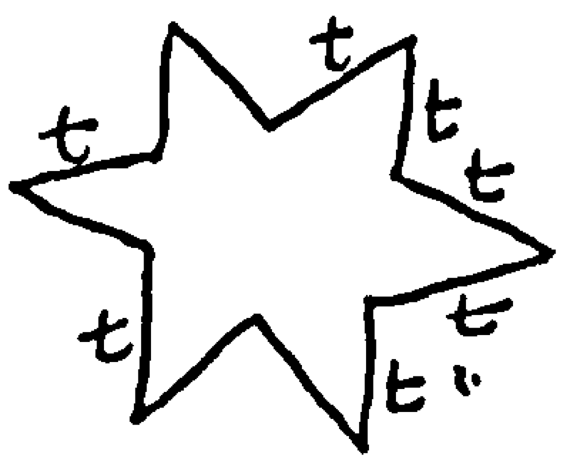


14. Shape:



All sides are of length m
There are 18 sides altogether

15. Shape:



All sides are of length t
There are 12 sides altogether

MACHINE-MATHS: SHEET 8.

	Instruction Pad	Print-out Pad
1.	Add 5 to n	
2.	Multiply x by 7	
3.	Add 4 to 3 times n	
4.	Add 5 to $n+2$	
5.	Multiply $3 \times a$ by 5	
6.	Multiply $a+4$ by 3	
7.	Add a 6 times	
8.	Add n 23 times	
9.	Multiply 4 by $x+2$	

MACHINE MATHS - SHEET 9.

	Instruction Pad	Print-out Pad
1.	Divide 8 by any number I give	$x \div 8$ $8 \div a$
2.	Add any number I give to 7, then multiply by 12.	$a+7\times 12$ $12\times a+7$ $(a+7)\times 12$ $12\times(a+7)$
3.	Take any number I give away from 60	$60-n$ $t-60$
4.	Multiply any number I give by itself	$a\times a$ $a\times b$ $x\times x$
5.	Add together any three numbers I give	$x+x+x$ $a+b+c$ $n+n+3$ $5+2+7$
6.	Multiply 4 by 5+n	$4\times 5+n$ $20+n$ $(5+n)\times 4$ $4\times(5+n)$
7.	$(5\times n)+(4\times n)$	$20\times n$ $9\times n$
8.	Add five times	$+5$ $+ \quad + \quad + \quad +$ $5+5+5+5+5$ $\times 5$
9.	Add 3 times a and 4 times b	$7\times a+b$ $7+a\times b$ $(3\times a)+(4\times b)$

MACHINE MATHS - SHEET 10.

	Instruction Pad	Print-out Pad
1.	$x+x+x+x$	
2.	$(4\times a)+(5\times a)$	
3.	$a+a+b+b+b+b$	
4.	$(2\times n)+(9\times n)$	
5.	$(3\times a)+(2\times b)$	
6.	$(4\times y)+6$	
7.	$5\times(t+4)$	
8.	$n+a+n+a+n$	
9.	$3\times 2\times a$	

10 cont.

	Instruction Pad	Print-out Pad
1.	$4+8+m$	
2.	$y+y+y+y+y+y$	
3.	$7+(c\times 2)$	
4.	$4\times m\times 3$	
5.	$(2\times n)+(6\times p)$	
6.	$(9\times y)+(6\times y)$	
7.	$(2\times a)+(3\times a)+(5\times a)$	
8.	$6+p+3$	
9.	$(6\times y)+(4\times a)+(2\times y)$	

APPENDIX 6

Pretest used in the small-scale teaching experiments described in Chapter 4, and in the class teaching (researcher) phase of the investigation (see Chapter 5).

The items included in the analysis are those circled, giving a total of 21 items.

The pretest was also used as the delayed posttest in both phases of the research referred to above.

However, note that this pretest was not administered to the first year group of the class teaching (researcher) phase of the study (see text, Chapter 5), nor was it used as delayed posttest for the first and third year groups in the same study (see Chapter 5, also Appendix 9).

When given the test as a pretest, the children were asked to complete the trial items first and check their answers against those given on the back page (not shown). Any queries relating to the meaning of these items were answered by the researcher-teacher. This step was omitted on the posttest, and children answered only the test items proper.

For an outline of which groups of items were suggested to provide an indication of understanding in each of the areas of difficulty identified by the research, see Chapter 5. Note, however, the caution given in that chapter against viewing a given item as measuring one conceptual area only.

Algebra 1

Name .... *School* .... *Class* ......

Date .... *Date of Birth* ....
day *month* *year*

Boy or Girl ....

Trial Item 1

What number does $a + 4$ stand for if $a = 2$ ....

What number does $4a$ stand for if $a = 2$ ....

Trial Item 2

Fill in the gaps:
(work down the page)

$x \longrightarrow 3x$	$x \longrightarrow x+3$	$x \longrightarrow 7x$	$x \longrightarrow x+8$
$2 \longrightarrow 6$	$5 \longrightarrow 8$	$2 \longrightarrow .$	$3 \longrightarrow .$
$5 \longrightarrow .$	$4 \longrightarrow .$		
	$n \longrightarrow .$		

Now check your answers against the answers on the back page.

---

1. Fill in the gaps:

$x \longrightarrow x + 2$	$x \longrightarrow 4x$
$6 \longrightarrow .$	$3 \longrightarrow .$
$r \longrightarrow .$	

---

2. Write down the smallest and the largest of these: smallest largest

$n + 1$, $n + 4$, $n - 3$, n, $n - 7$. ....

---

3. Which is the larger, $2n$ or $n + 2$? ....

Explain: ....

---

4. 4 added to n can be written as $n + 4$.
Add 4 onto each of these:

8 $n + 5$ $3n$

....

n multiplied by 4 can be written as $4n$.
Multiply each of these by 4:

8 $n + 5$ $3n$

....

---

5. If $a + b = 43$

$a + b + 2 =$ ......

If $n - 246 = 762$

$n - 247 =$ ......

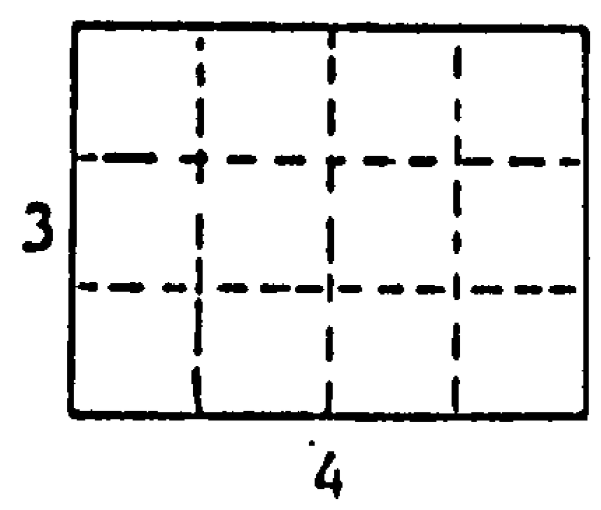
If $e + f = 8$

$e + f + g =$ ......

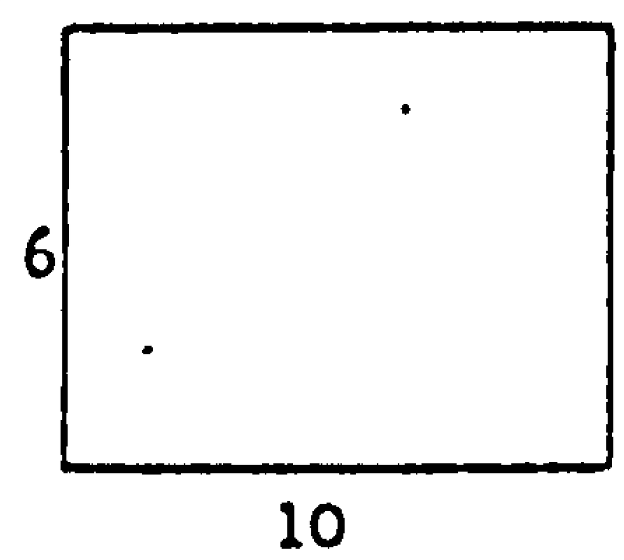
6. What can you say about a if $a + 5 = 8$ ....

What can you say about b if $b + 2$ is equal to $2b$ ....

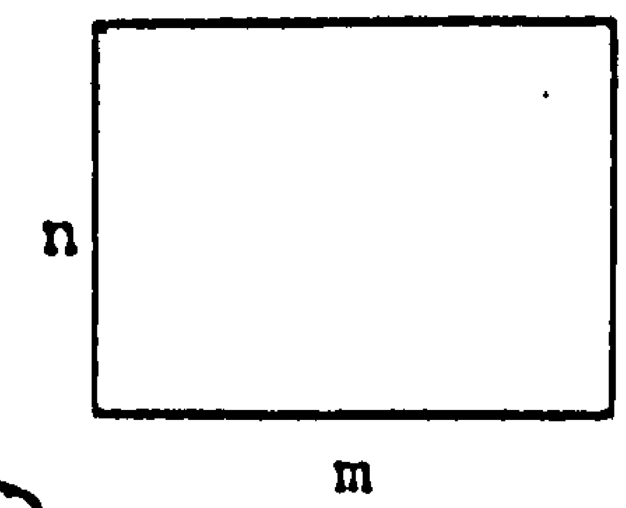
7. What are the areas of these shapes?



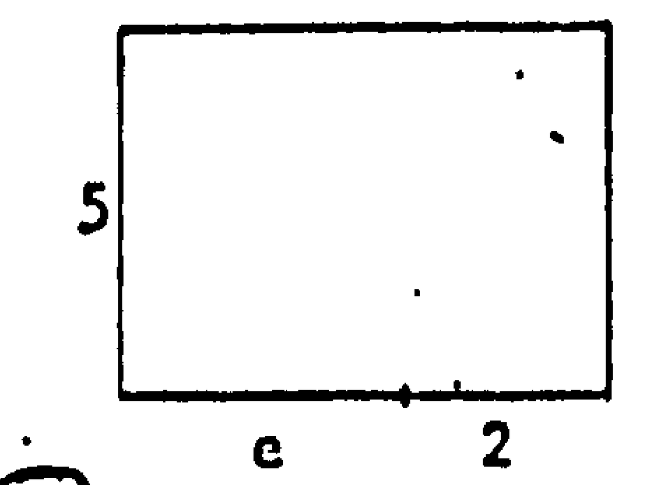
A = ...



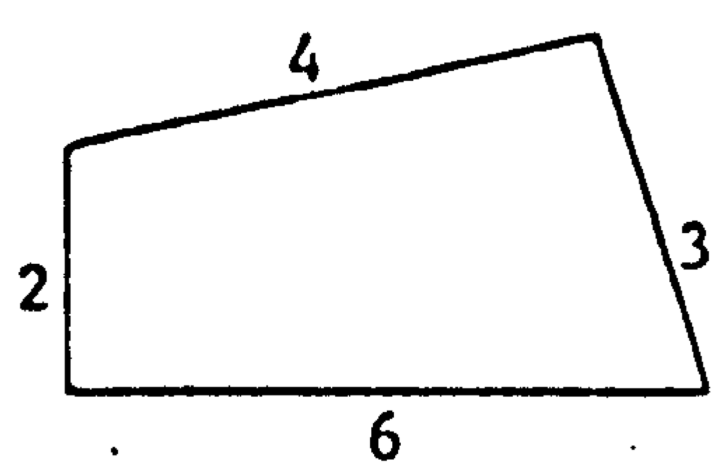
A = ...



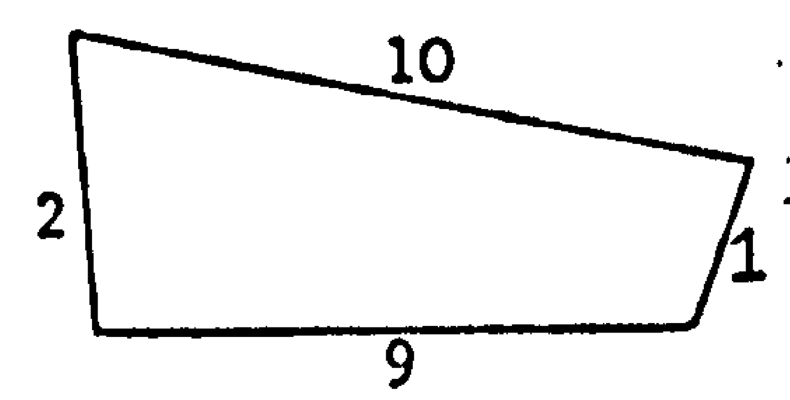
A = ...



A = ...



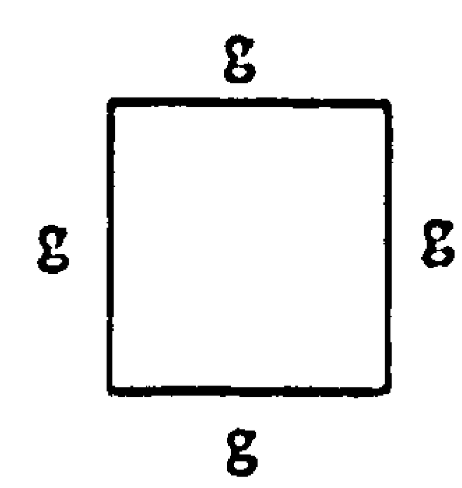
8. The perimeter of this shape is equal to $6 + 3 + 4 + 2$, which equals 15.



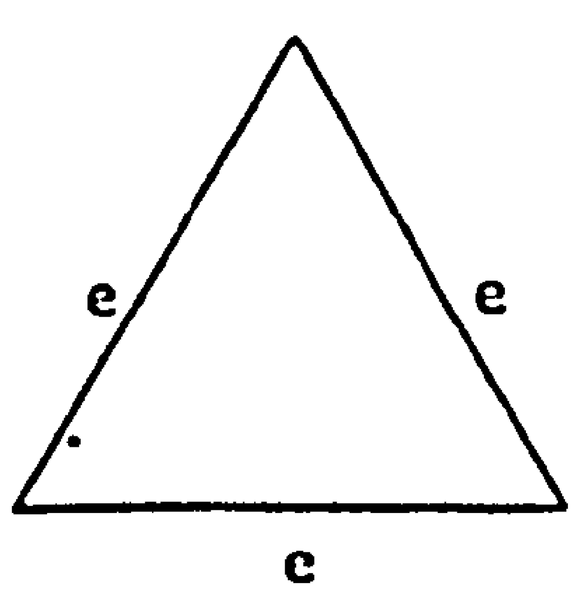
Work out the perimeter of this shape. p = ...

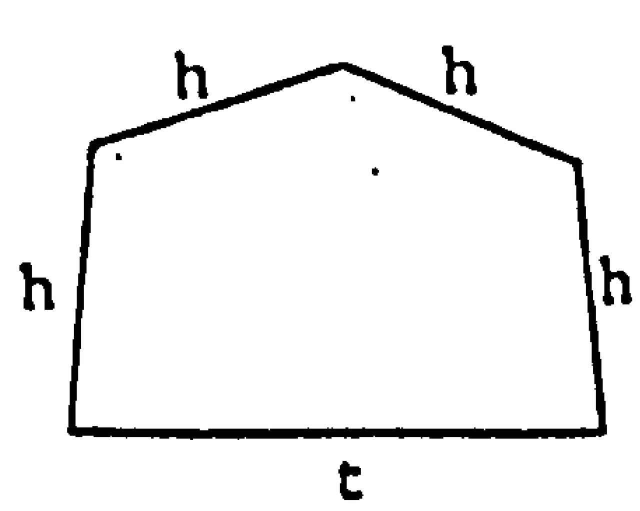
9.

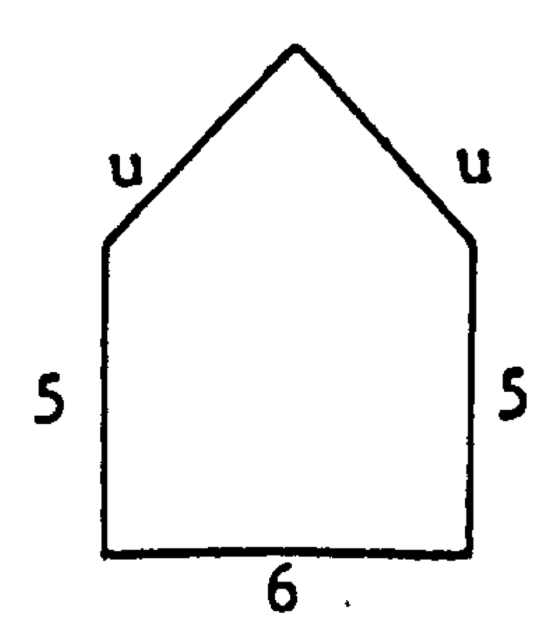
This square has sides of length g.
So, for its perimeter, we can write $p = 4g$.



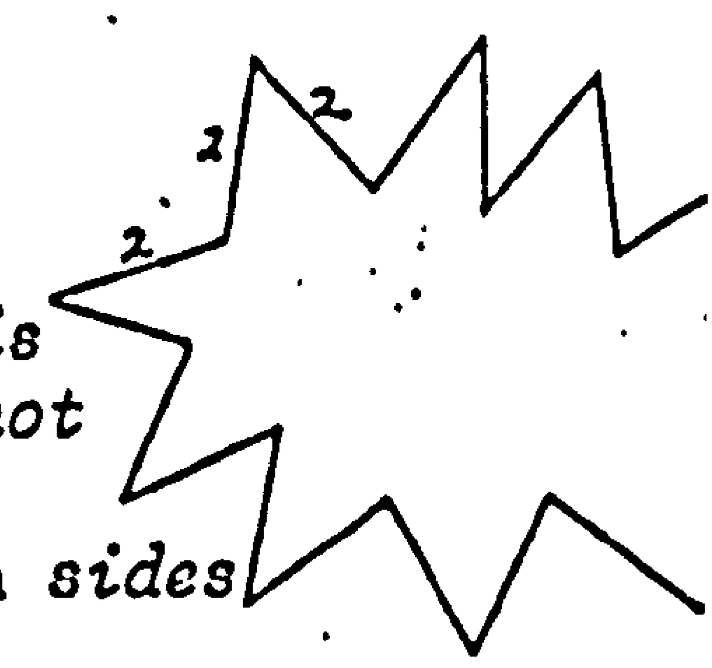
What can we write for the perimeter of each of these shapes?







Part of this figure is not drawn.
There are n *sides altogether,*
all of length 2.



p = ... p = ... p = ... p = ...

1 What can you say about u if $u = v + 3$ and $v = 1$ ....

2 What can you say about m if $m = 3n + 1$ and $n = 4$ ....

12. If John has J marbles and Peter has P marbles, what could you write for the number of marbles they have altogether? ....

---

13. a + 3a can be written more simply as 4a.

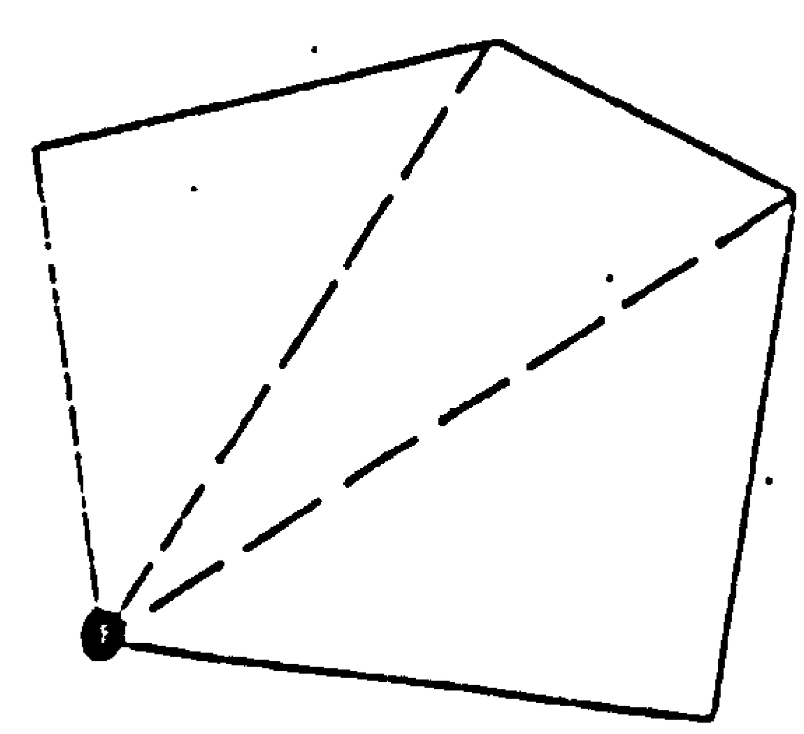
Write these more simply, where possible:

2a + 5a =	....		
2a + 5b =	....	3a − (b + a) =	....
(a + b) + a =	....	a + 4 + a − 4 =	....
2a + 5b + a =	....	3a − b + a =	....
(a − b) + b =	....	(a + b) + (a − b) =	....

---

14. What can you say about r if r = s + t and r + s + t = 30 ....

15.



In a shape like this you can work out the number of diagonals by taking away 3 from the number of sides.

So, a shape with 5 sides has 2 diagonals;

a shape with 57 sides has .... diagonals;

a shape with k sides has .... diagonals.

---

16. What can you say about c if c + d = 10 and c is less than d ....

---

17. When are the following true – always, never, or sometimes?
Underline the correct answer:

A + B + C = C + A + B Always. Never. Sometimes, when ....

L + M + N = L + P + N Always. Never. Sometimes, when ....

APPENDIX 7

Immediate posttest used in the small-scale teaching experiments described in Chapter 4, and in the class teaching (researcher) phase of the investigation (see Chapter 5), where it was used with the second and fourth year groups only (the immediate posttest used with the first and third year groups is given in Appendix 9).

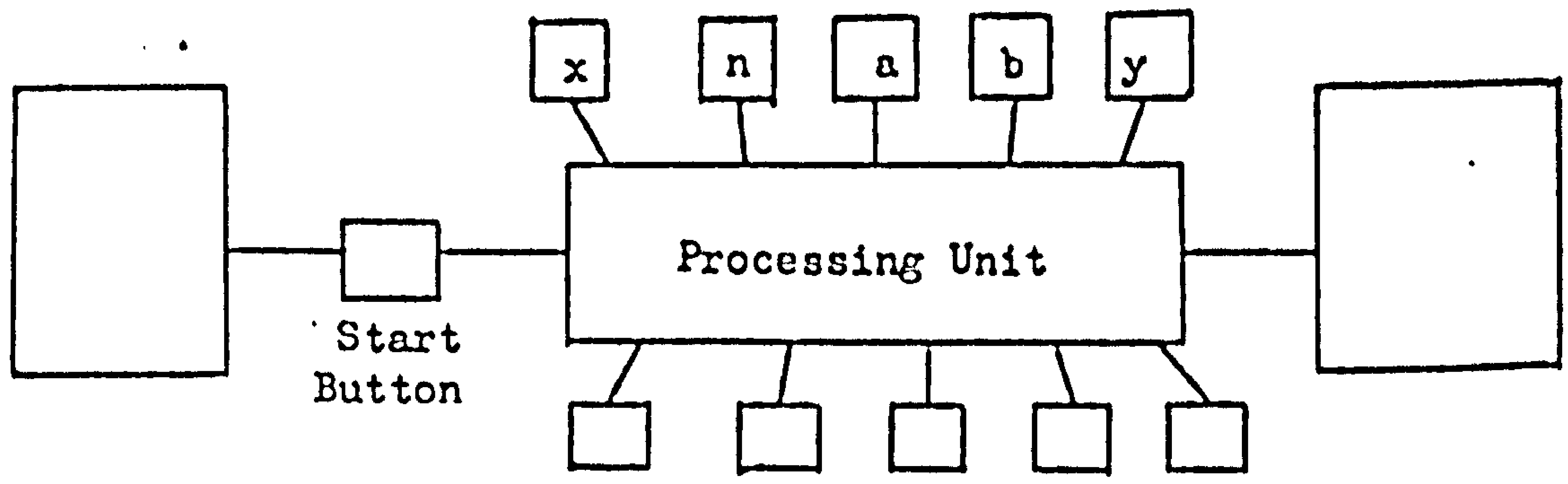
The items included in the analysis are those circled, giving a total of 21 items.

For an outline of which groups of items were suggested to provide an indication of understanding in each of the areas of difficulty identified by the research, see Chapter 5. Note, however, the caution given in that chapter against viewing a given item as measuring one conceptual area only.

SESM: MACHINE MATHS

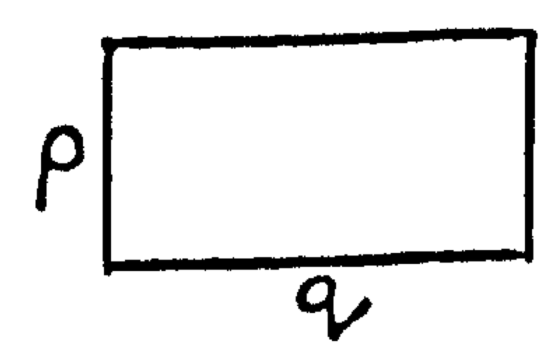
Name: School:

Date:

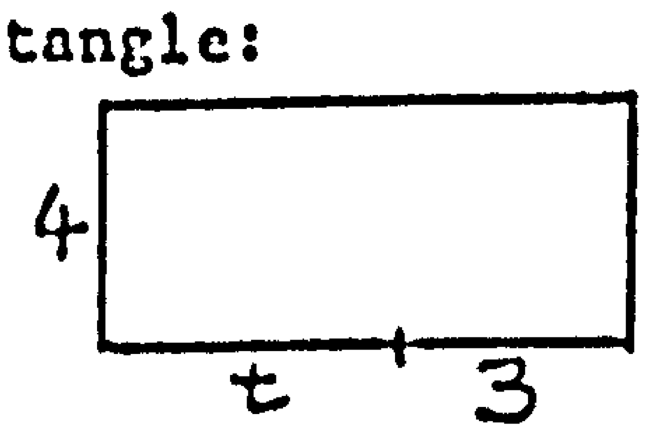


---

1. What can you write for the area of this rectangle:



2. What can you write for the area of this rectangle:



3. Add 3 onto 4a ....

4. Multiply by 3: a+5 ....

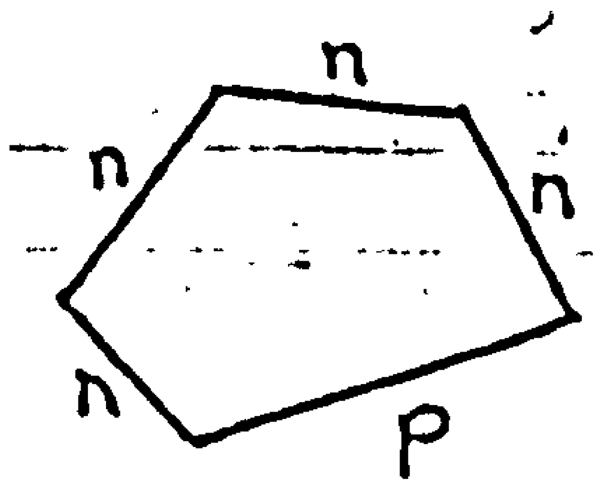
5. Multiply by 4: 3a ....

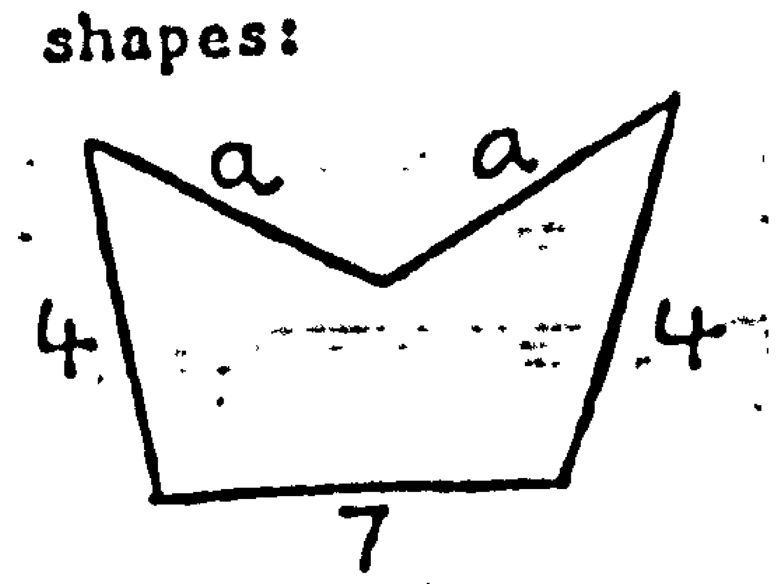
6. If p+q = 6

   p+q+r = ...

7. I have c records and my sister has d records. How many records do we have altogether? ....

8. What can you write for the perimeter of these shapes:





Perimeter = ... Perimeter = ...

9. Find the perimeter of a shape which has n sides which are all 4cm. long.

Perimeter = ...

10. If a = b+2
and b = 1
a = ...

If t = 3y +.1
and y = 5
t = ...

11. Write more simply where possible:

2a + 7a = ...

2a + 7b = ...

(a+b) + a = ...

2a + 7b + a = ...

5a - b + a = ...

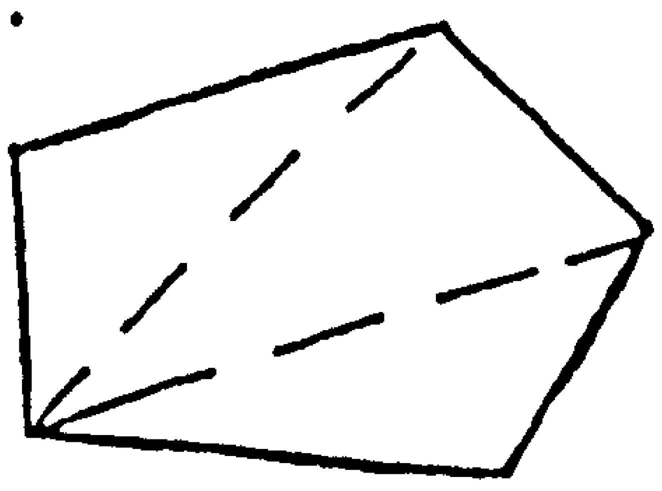
(a - b) + b = ...

12. If c + d = 8
and c is less than d
what can you say about c? ....

13. a + b + c = a + m + c

When is this true - Always
Never
Sometimes, when ....

14. In a shape like this you can work out how many diagonals it has by taking away 3 from the number of sides.



How many diagonals has a shape with
p sides ....

15. What can you say about e if
e = f + g
and e + f + g = 30
(Ans) e = ...

APPENDIX 8

Paper and pencil test designed to investigate various algebraic notation concepts among secondary school children at second to fifth year level (ages 13 to 16). The test includes items relating to the use of brackets and to conjoining in algebraic addition.

SESM ALGEBRA

Name: .... School: ....

Class: ....

1. What does 5y mean? Tick every answer you think is correct:

5+y ....

5 and y ....

5×y ....

5+5+5+5+5 ....

y+y+y+y+y ....

other answer (please write) ....

2. What does e3 mean? Tick every answer you think is correct:

e×3 ....

e+3 ....

3×e ....

3+3+3 ....

e+e+e ....

e and 3 ....

other answer (please write) ....

3. What does yz mean? Tick every answer you think is correct:

y and z ....

y×z ....

y+z ....

25+26 ....

25×26 ....

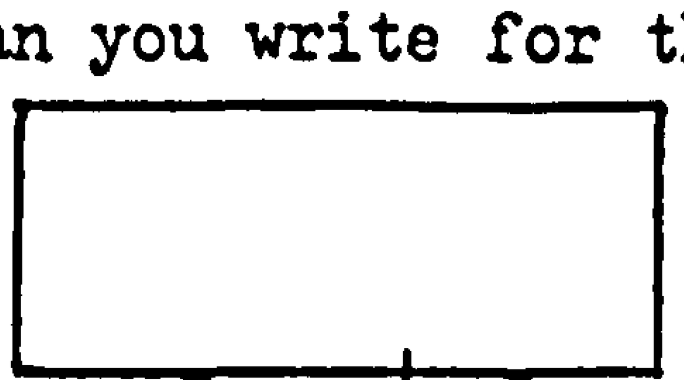
other answer (please write) ....

4. What can you write for 3 added to 5y? ....

5. What can you write for m+5 multiplied by 3? ....

6. What can you write for the area of this rectangle:

6



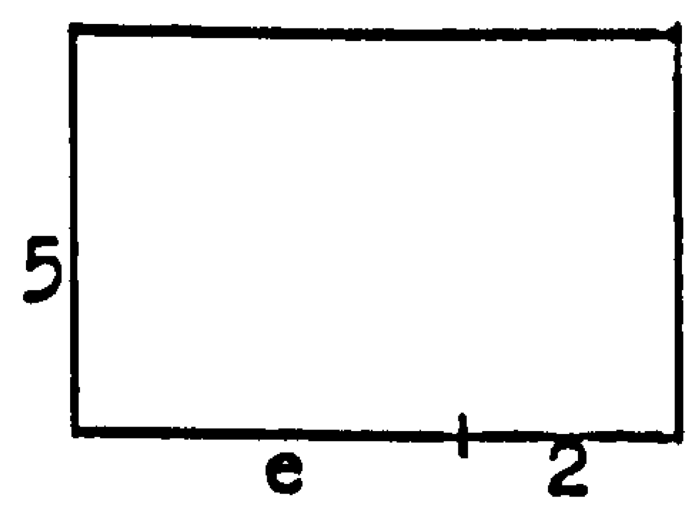
p 2

Area: ....

7. If a=3, what does 4a equal? ....

8. If b=5, what does b7 equal? ....

9. Which of the following expressions can you write for the area of this rectangle? Tick <u>every</u> one you think is correct:



5×e+2 ....
5×(e+2) ....
10e ....
5×e2 ....
5(e+2) ....
e+2×5 ....
none correct ....

10. Which of the following expressions can you write for 2 added to 5a? Tick <u>every</u> one you think is correct:

5a add 2 ....
5a+2 ....
7a ....
2+5a ....
7 ....
10a ....
(2+5)a ....
none correct ....

11. Which of the following expressions can you write for e+2 multiplied by 3? Tick <u>every</u> one you think is correct:

e+6 ....
3x(e+2) ....
3xe2 ....
3(e+2) ....
3e+6 ....
e+2x3 ....
none correct ....

12. Write more simply if you can. If you think it <u>cannot</u> be written more simply, write NO.

2a+5b+3a ....
4+3y ....
3a+7b ....
a+a+3b+5a ....
5y-2t ....

APPENDIX 9

Immediate and delayed posttests used with the first and third year groups in the class teaching (researcher) phase of the investigation (see Chapter 5). These tests comprise shortened versions of the tests used with the second and fourth year groups in the same study (see text, Chapter 5). The items omitted from the latter tests are those concerned primarily with simplifying algebraic expressions.

The items in the analysis are those circled, giving a total of 16 items.

For an outline of which groups of items were suggested to provide an indication of understanding in each of the areas of difficulty identified by the research, see Chapter 5. Note, however, the caution given in that chapter against viewing a given item as measuring one conceptual area only.

IMMEDIATE POST-TEST (FIRST AND THIRD YEARS)

SESM: MACHINE MATHS

Name: .... School: ....

Date: .... Class: ....

1. Write the instructions for adding 5 to n ....

2. Write the instruction for multiplying y by 7 ....

3. Write the instructions for adding a number to itself ....

4. Add 3 to 4a ....

5. Multiply a+5 by 3 ....

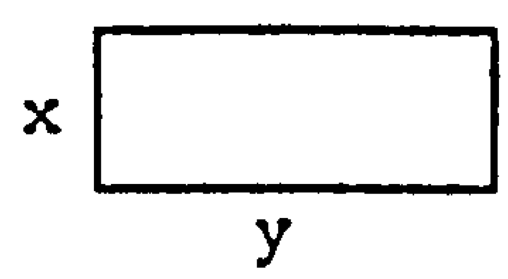
6. Multiply 3xa by 4 ....

7. If p+q = 6, then p+q+r = ...

8. I have c records and my sister has d records. What can you write for how many records we have altogether?

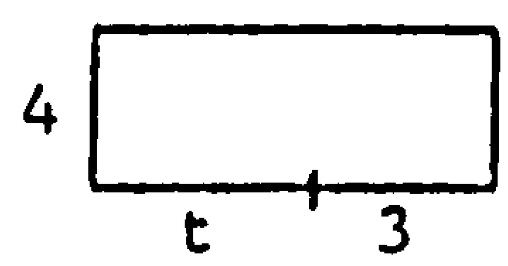
....

9. What can you write for the area of this rectangle:



Area: ....

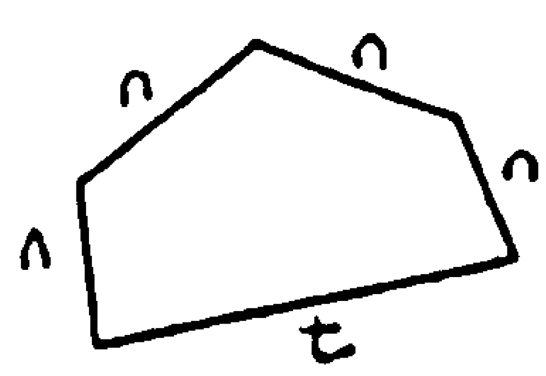
10. What can you write for the area of this rectangle:

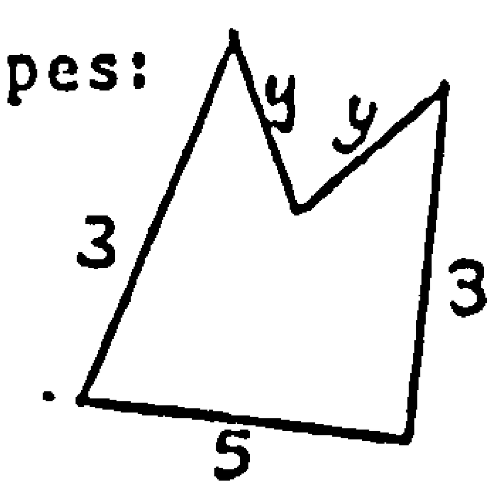


Area: ....

11. What can you write for the perimeter of a square ....

12. What can you write for the perimeter of these shapes:





Perimeter: .... Perimeter: ....

13. What can you write for the perimeter of a shape which has n sides which are all 4cm. long

....

14. If a = b+2 and b = 3, then a = ...

(15) If $t = (3xy)+1$ and $y = 5$, then $t =$ ....

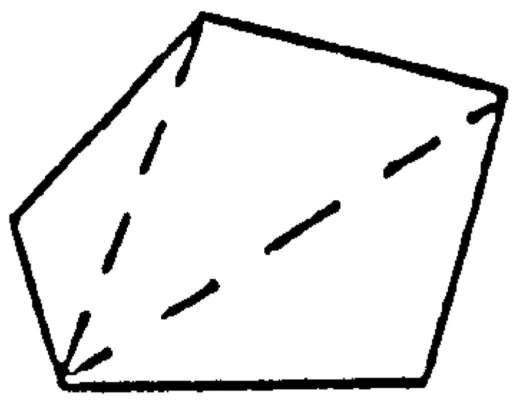
(16) If $c+d = 8$ and c is less than d, what can you say about c?

....

(17) If $e = f+g$ and $e+f+g = 30$, what can you say about e?

....

(18) In a shape like this you can work out how many diagonals it has by <u>taking away 3</u> from the number of sides



How many diagonals has a shape with p sides ....

(19) $a+b+c = a+m+c$

When is this true:

always ....

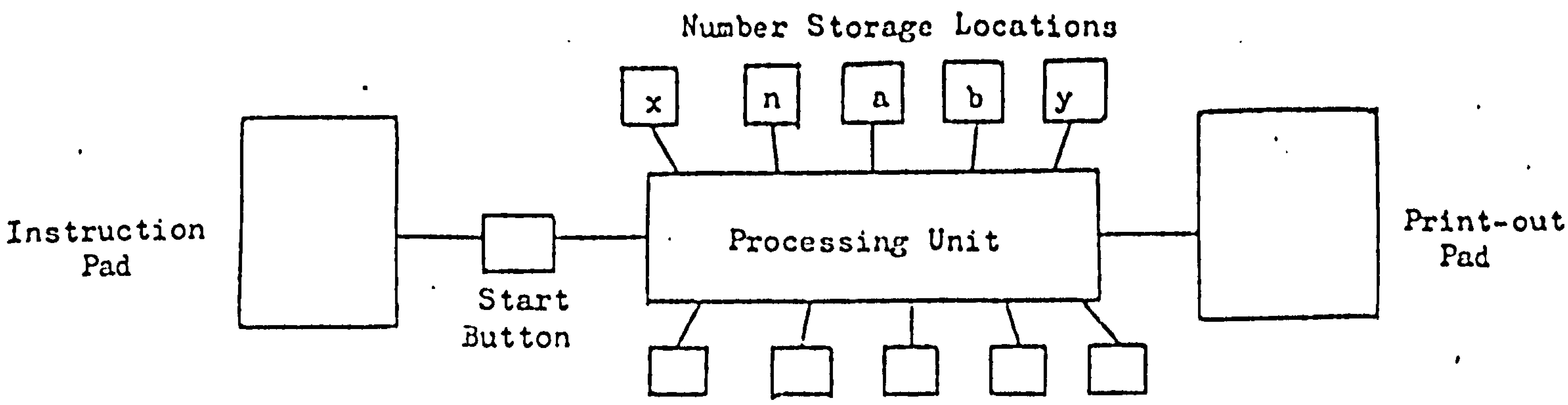
never ....

sometimes, when ....

Algebra 3

Name: School:

Date:



1. Write the instructions for adding 5 to a ....

2. Write the instructions for multiplying y by 4 ....

3. Write the instructions for adding a number to itself ....

4. Add m to 7 and then multiply by 3 ....

5. Add 4 onto 3a ....

6. Multiply by 4: a+5 ....

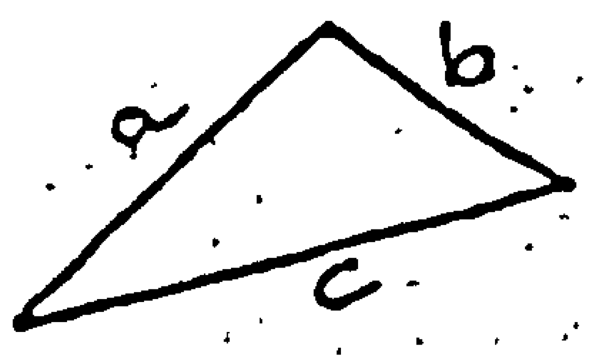
7. Multiply by 3: 4a ....

8. If a + b = 8

   a + b + c= ....

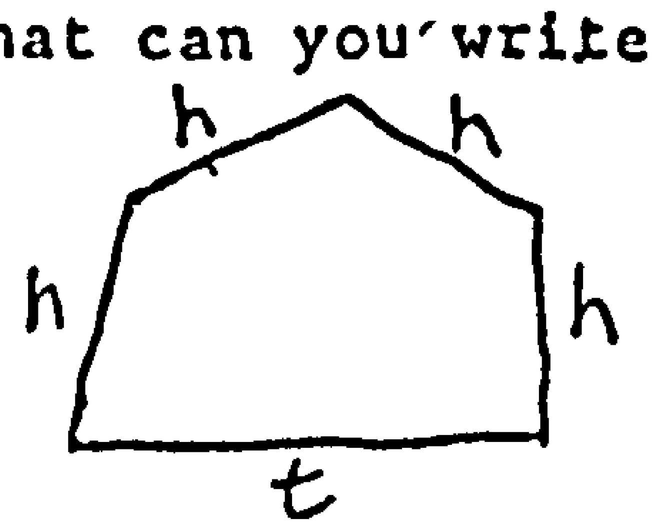
9. I have n sweets and my sister has y sweets. What can you write for how many sweets we have altogether ....

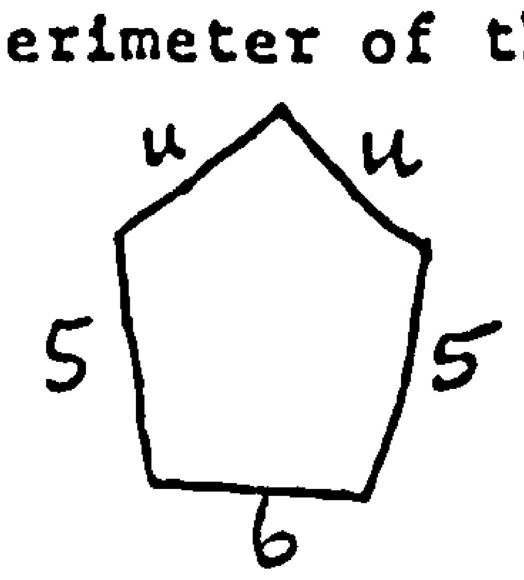
10. What can you write for the perimeter of this triangle:



Perimeter: ....

11. What can you write for the perimeter of these shapes:





Perimeter = Perimeter =

12. Find the perimeter of a shape which has p sides which are all 4cm. long.

Perimeter = ...

13. If a = b + c and b = 1, then a = ...

14. If m = (3 x n) + 1 and n = 4, then m = ...

15. What can you say about r if

$$r = s + t$$

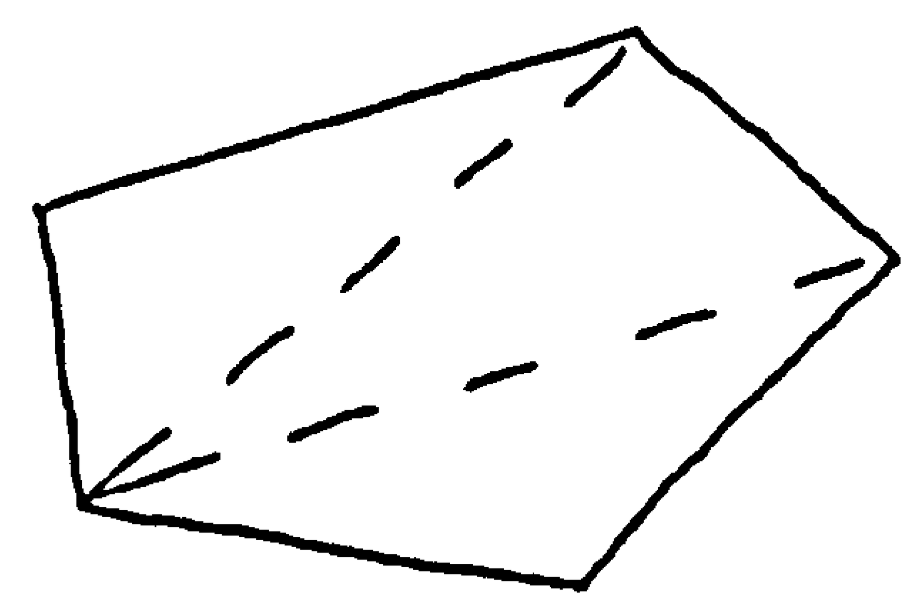
and $r + s + t = 30$

(Ans) r = ...

16. What can you say about c if c + d = 10 and c is less than d?

....

17. In a shape like this you can work out how many diagonals it has by taking away 3 from the number of sides.



so a shape with 5 sides has 2 diagonals.

How many diagonals has a shape with K sides ....

18. a + m + y = a + p + y

When is this true (tick the correct answer)

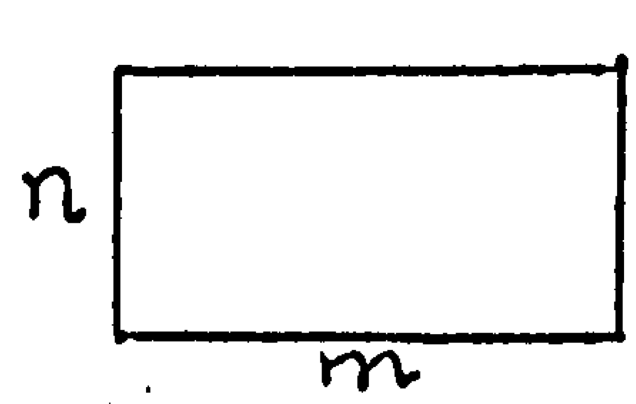
Always ....

Never ....

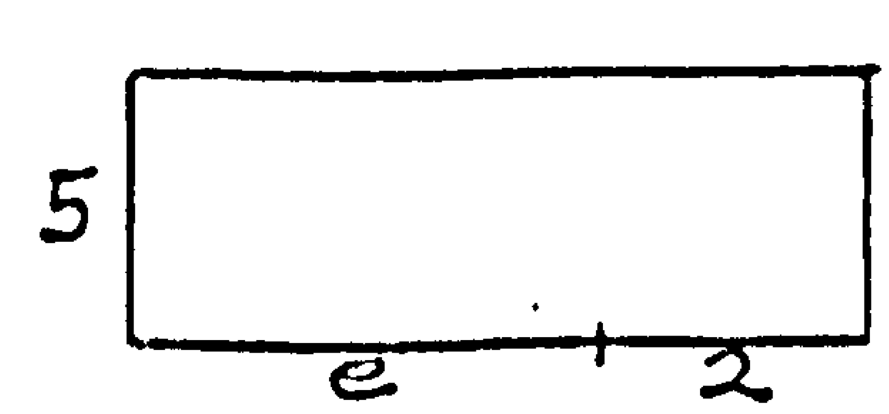
Sometimes ....

If 'Sometimes', say when it would be true ....

19. What can you write for the area of these rectangles:



Area=



Area=

APPENDIX 10

Amended pre- and immediate posttests for use in class teaching (other teachers) study (see Chapter 5). The pretest is also used as the delayed posttest.

For an outline of which groups of items were suggested to provide an indication of understanding in each of the areas of difficulty identified by the research, see Chapter 5. Note, however, the caution given in that chapter against viewing a given item as measuring one conceptual area only.

Algebra 1

Name .... *School* .... *Class* ......

Date .... *Date of Birth* ....
day month year

Boy or Girl ....

Trial Item 1

What number does $a + 4$ stand for if $a = 2$ ....

What number does $4a$ stand for if $a = 2$ ....

Trial Item 2

Fill in the gaps:
(work down the page)

$x \longrightarrow 3x$	$x \longrightarrow x+3$	$x \longrightarrow 7x$	$x \longrightarrow x+8$
$2 \longrightarrow 6$	$5 \longrightarrow 8$	$2 \longrightarrow .$	$3 \longrightarrow .$
$5 \longrightarrow .$	$4 \longrightarrow .$		
	$n \longrightarrow .$		

Now check your answers against the answers on the back page.

1\. Fill in the gaps:

$x \longrightarrow x + 2$	$x \longrightarrow 4x$
$6 \longrightarrow .$	$3 \longrightarrow .$
$r \longrightarrow .$	

2\. Write down the smallest and the largest of these: smallest largest

$n + 1$, $n + 4$, $n - 3$, n, $n - 7$. ....

3\. Which is the larger, $2n$ or $n + 2$? ....

Explain: ....

4\. <u>4 added to n</u> can be written as <u>n + 4</u>.
Add 4 onto each of these:

8 $n + 5$ $3n$

....

<u>n multiplied by 4</u> can be written as <u>4n</u>.
Multiply each of these by 4:

8 $n + 5$ $3n$

....

5\. If $a + b = 43$

$a + b + 2 =$ ......

If $n - 246 = 762$

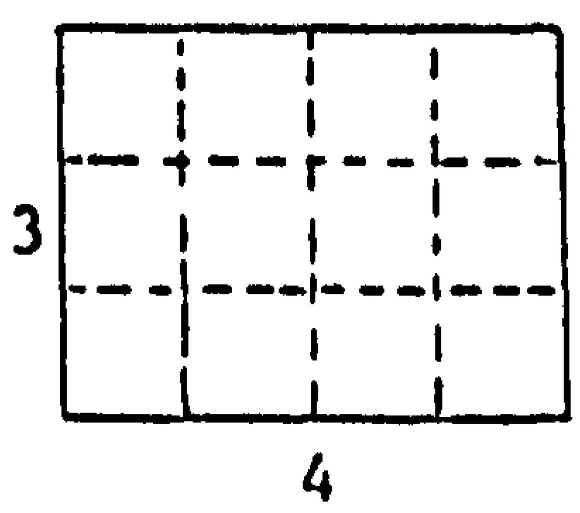
$n - 247 =$ ......

If $e + f = 8$

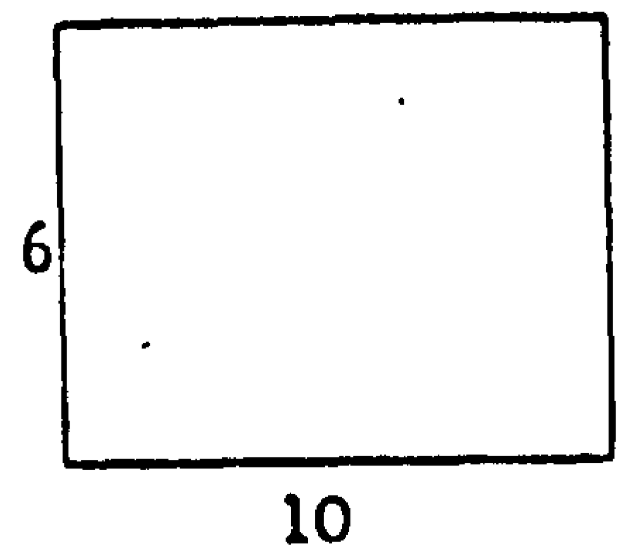
$e + f + g =$ ......

6. What can you say about a if $a + 5 = 8$ ....

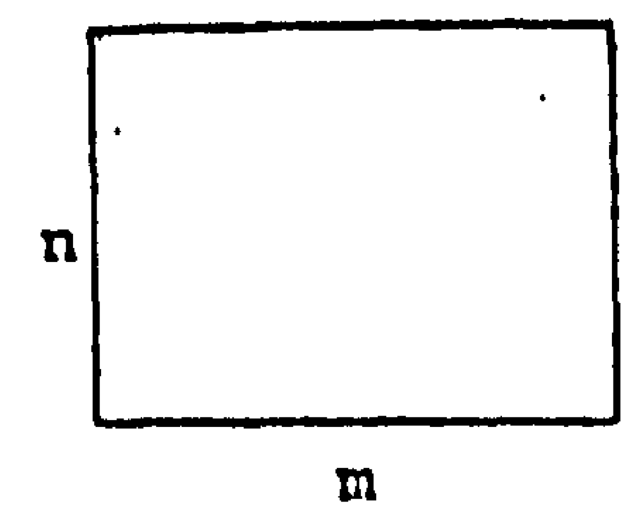
7. What are the areas of these shapes?



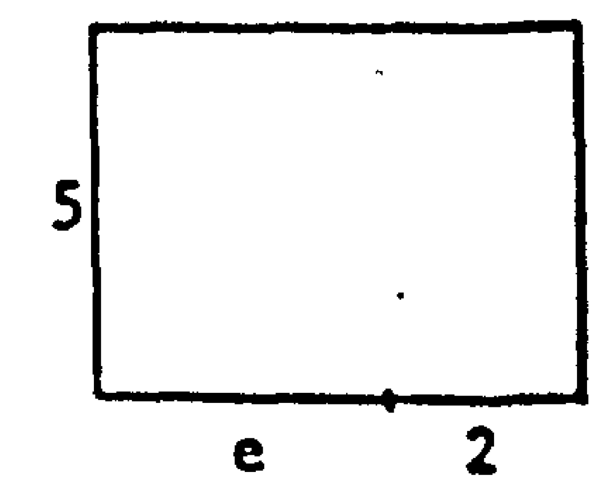
A = ...



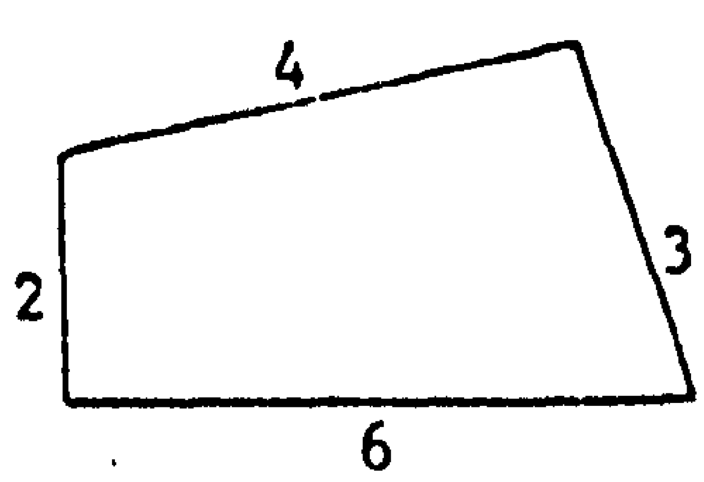
A = ...



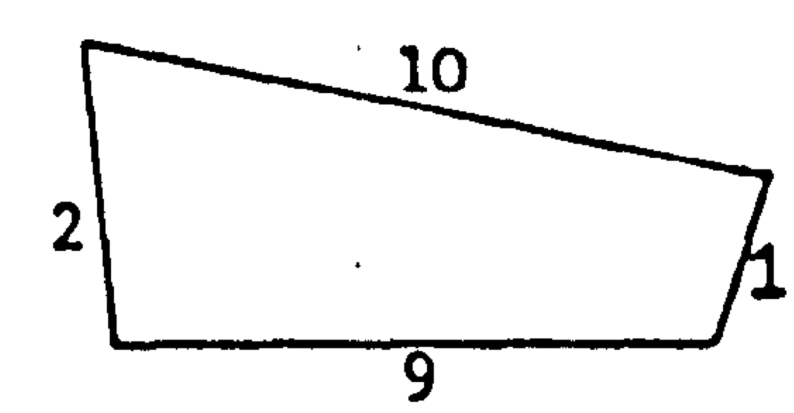
A = ...



A = ...

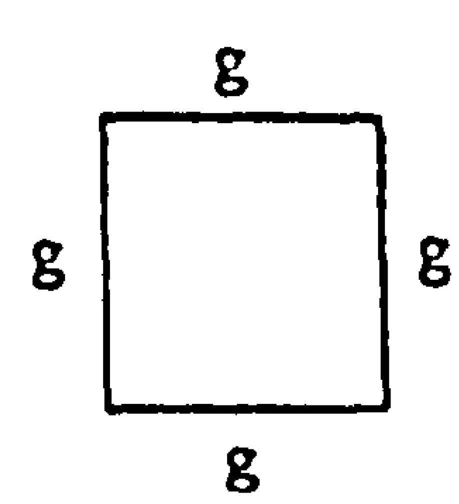


8. The perimeter of this shape is equal to $6 + 3 + 4 + 2$, which equals 15.



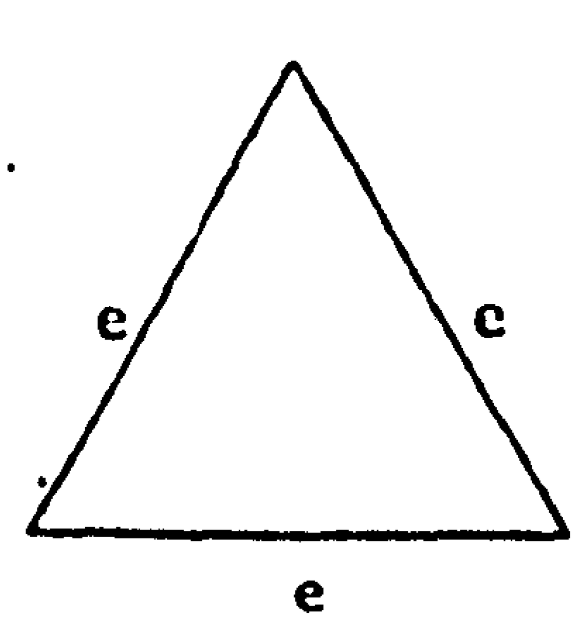
Work out the perimeter of this shape. p = ...

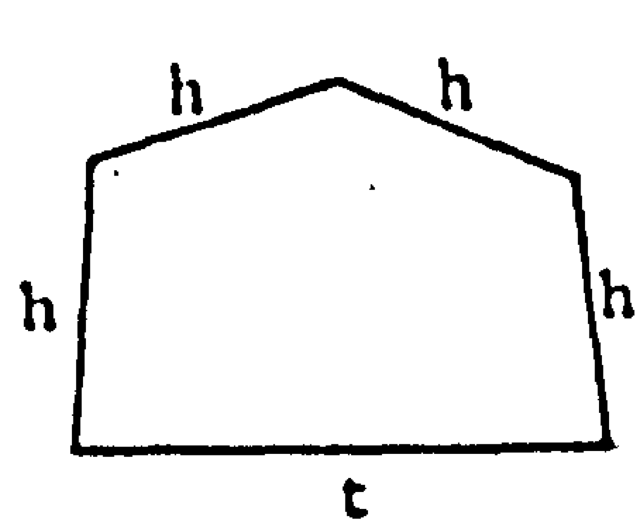
9.

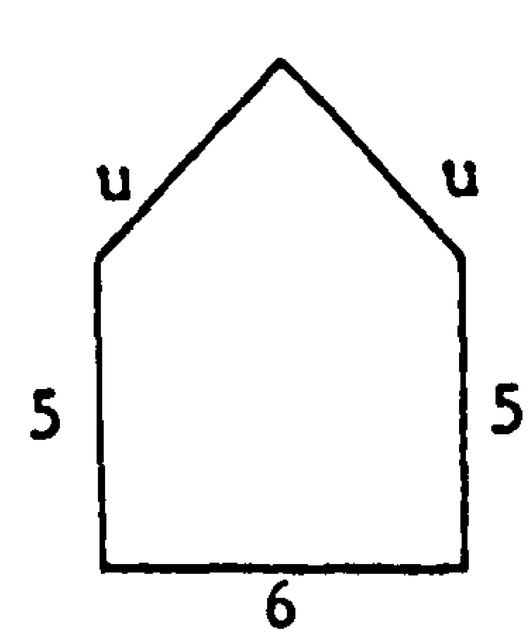


This square has sides of length g.
So, for its perimeter, we can write $p = 4g$.

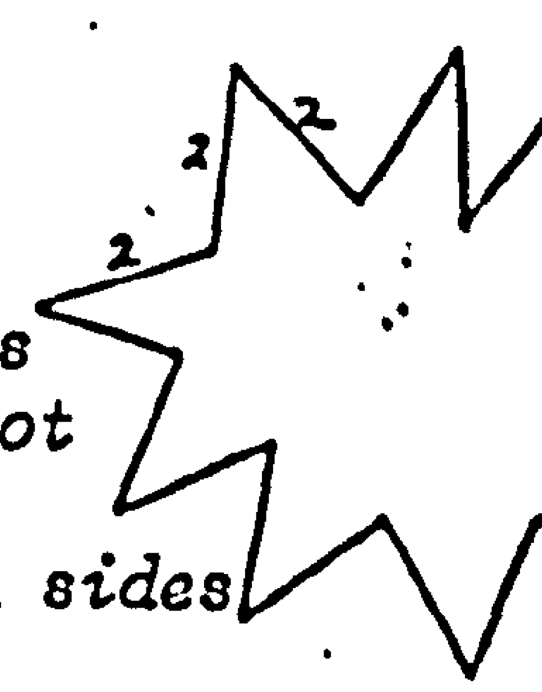
What can we write for the perimeter of each of these shapes?







Part of this figure is not drawn.
There are n *sides altogether, all of length 2.*



p = ...

p = ...

p = ...

p = ...

11. What can you say about u if $u = v + 3$ and $v = 1$ ....

What can you say about m if $m = 3n + 1$ and $n = 4$ ....

12. If John has J marbles and Peter has P marbles, what could you write for the number of marbles they have altogether? ....

---

13. a + 3a can be written more simply as 4a.

Write these more simply, where possible:

$2a + 5a =$ ....

$2a + 5b =$ ....

$(a + b) + a =$ ....

$2a + 5b + a =$ ....

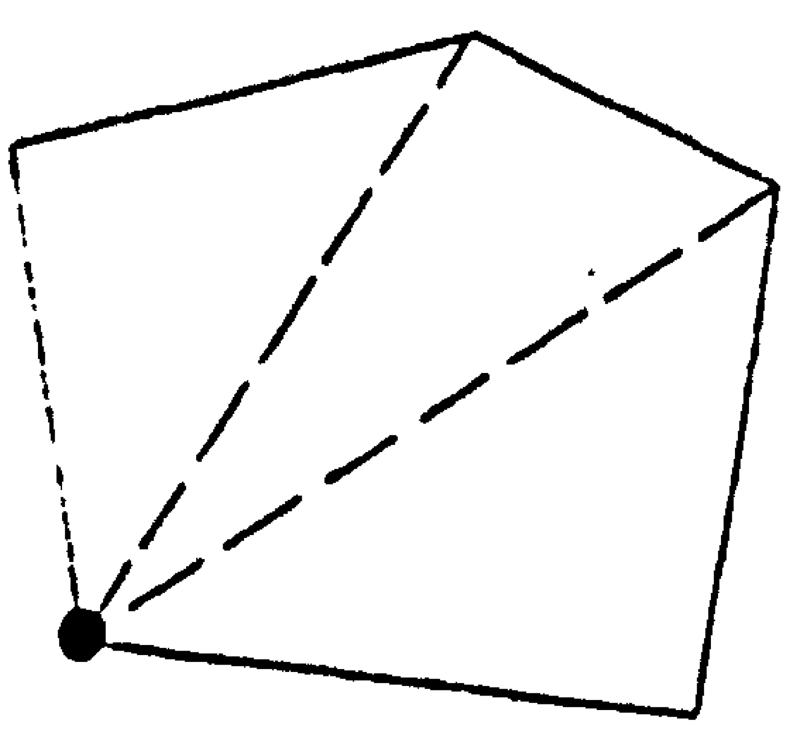
$3a - b + a =$ ....

---

14. What can you say about r if $r = s + t$

and $r + s + t = 30$ ....

15.



In a shape like this you can work out the number of diagonals by taking away 3 from the number of sides.

So, a shape with 5 sides has 2 diagonals;

a shape with 57 sides has .... diagonals;

a shape with k sides has .... diagonals.

---

16. What can you say about c if $c + d = 10$

and c is less than d ....

---

17. When are the following true - always, never, or sometimes?
Underline the correct answer:

$A + B + C = C + A + B$ Always. Never. Sometimes, when ....

$L + M + N = L + P + N$ Always. Never. Sometimes, when ....

18. Which of the following expressions can you write for e+2 multiplied by 3?
Tick every one you think is correct:

e+6 ....

$3\times(e+2)$ ....

$3\times e2$ ....

3(e+2) ....

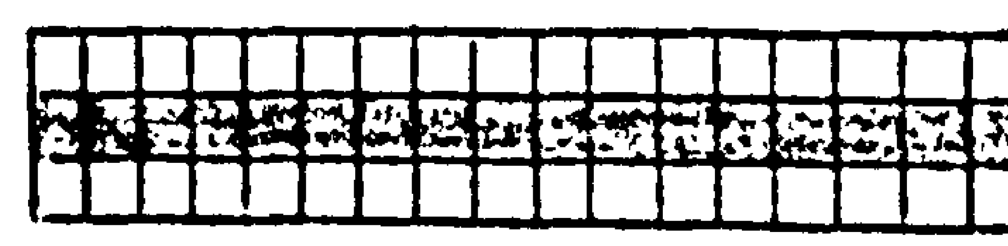
3e+6 ....

$e+2\times 3$ ....

None correct ....

19. I put grey squares in a row like this: 

Then I put a row of white squares along each side to match, like this:



What can you write for the number of white squares I will need if I use n grey squares? ....

How many white squares will I need if I use 50 grey squares? ....

Trial Item

ANSWERS →

What number does a + 4 stand for if a = 2 ..6...

What number does 4a stand for if a = 2 ..8...

$x \longrightarrow 3x$
$2 \longrightarrow 6$
$5 \longrightarrow 15$

$x \longrightarrow x+3$
$5 \longrightarrow 8$
$4 \longrightarrow 7$
$n \longrightarrow n+3$

$x \longrightarrow 7x$
$2 \longrightarrow 14$

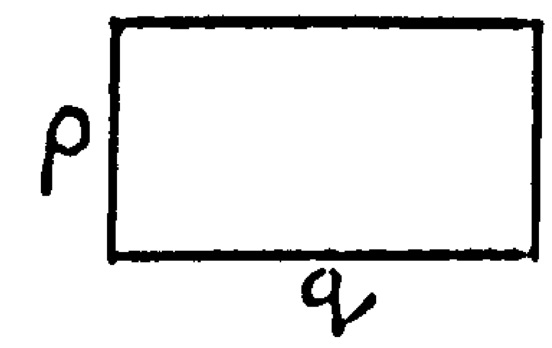
$x \longrightarrow x+8$
$3 \longrightarrow 11$

SESM: MACHINE MATHS

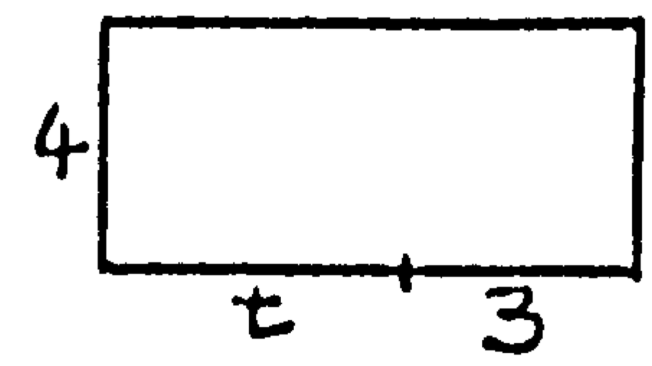
Name: School:

Date: Class:

1. What can you write for the area of this rectangle:



2. What can you write for the area of this rectangle:



3. Add 3 onto 4a ....

4. Multiply by 3: a+5 ....

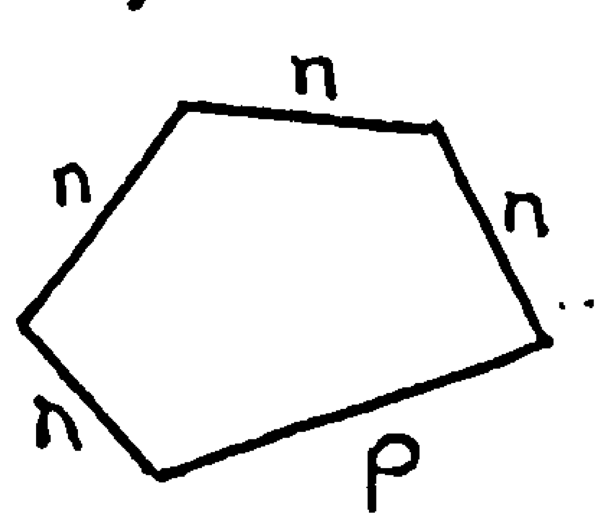
5. Multiply by 4: 3a ....

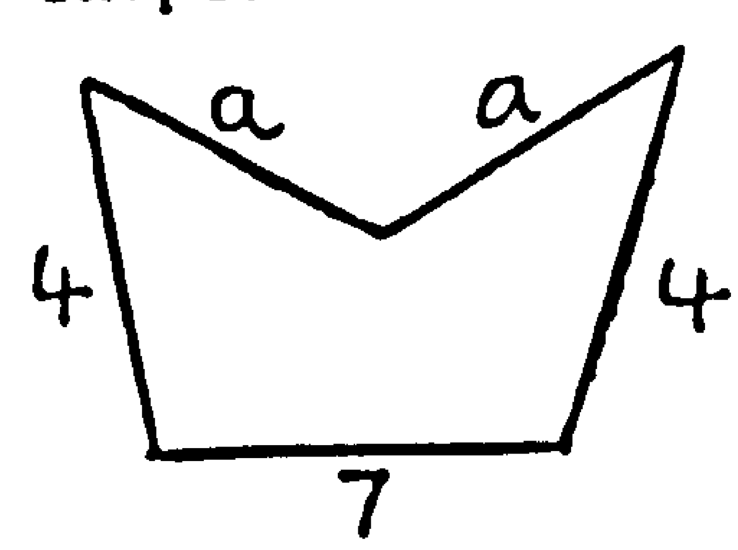
6. If p+q = 6

   p+q+r = ...

7. I have c records and my sister has d records. How many records do we have altogether? ....

8. What can you write for the perimeter of these shapes:





Perimeter = ... Perimeter = ...

9. Find the perimeter of a shape which has n sides which are all 4cm. long.

   Perimeter = ...

10. If a = b+2 If t = 3y + 1
    and b = 1 and y = 5
    a = ... t = ...

11. Write more simply where possible:

2a + 7a = ...

2a + 7b = ...

(a+b) + a = ...

2a + 7b + a = ...

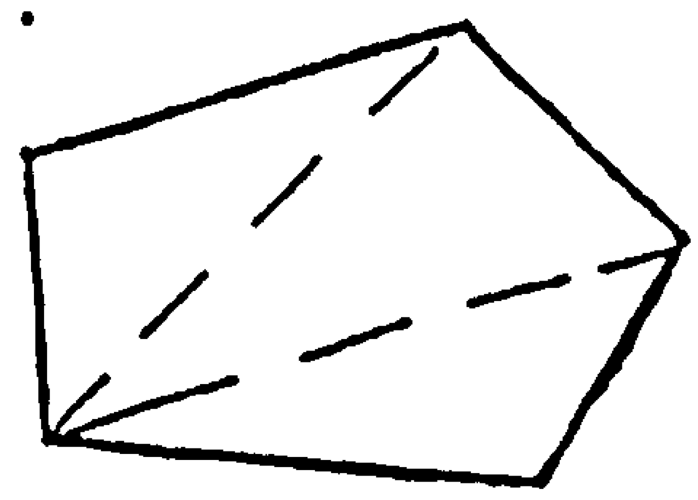
5a - b + a = ...

12. If c + d = 8
and c is less than d
what can you say about c? ....

13. a + b + c = a + m + c

When is this true - Always
Never
Sometimes, when ....

14. In a shape like this you can work out how many diagonals it has by <u>taking away 3</u> from the number of sides.



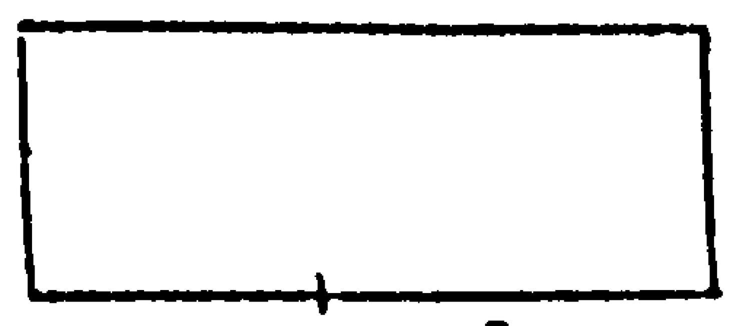
How many diagonals has a shape with
p sides ....

15. What can you say about e if

e = f + g
and e + f + g = 30

(Ans) e = ...

16. Which of the following expressions can you write for the area of this rectangle? Tick <u>every</u> one you think is correct:



3×a+5 ....

3×(a+5)..............

15a ....

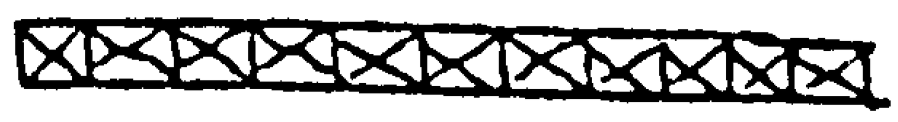
3×a5 ....

3(a+5) ....

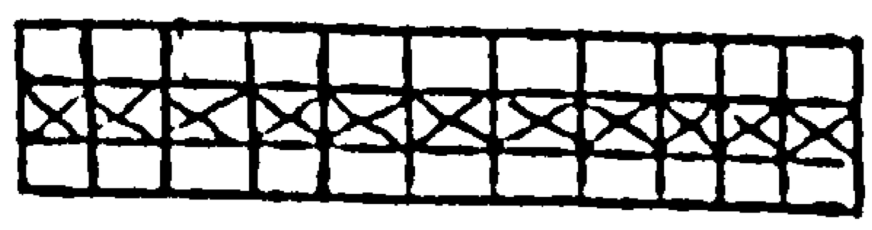
a+5×3 ....

None correct ....

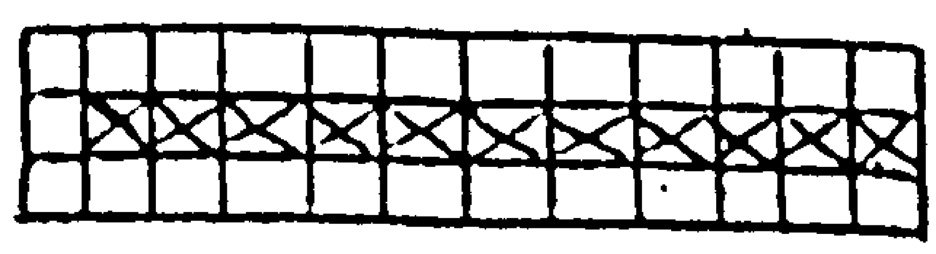
17. I put 'crossed' squares in a row like this:



Then I put a row of plain squares along each side to match:



Then I put 3 plain squares across the end:



What can you write for the number of <u>plain</u> squares I will need if I use n 'crossed' squares? ....

How many plain squares will I need if I use 50 'crossed' squares? ....

APPENDIX 11

Outline of the teaching programme used with the control class in School A (see Chapter 5, 'Class Teaching (Other Teachers)').

The outline is organised as follows:

1. Brief notes on the approach followed by the two textbooks usually used to introduce algebra in School A.

2. Examples of the exercises used.

Notes on the Control Class Teaching Programme Used in School A

The two text books generally used with the first year classes in school A are 'Headway Maths 2' by Roy Hollands and Howell Moses, and 'Maths to 16(1)' by Ronald Bolt and Charles Reynolds, although these are often supplemented with additional examples on the board or on worksheets and by extra teacher explanation.

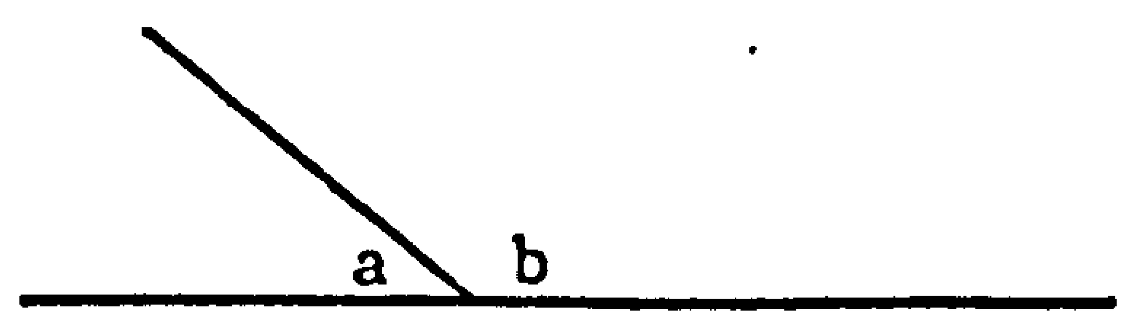
'Maths to 16(1)'

The relevant chapter in 'Maths to 16(1)' is chapter 17 on 'Letters for Numbers'. However, in earlier chapters in this book, some algebra is introduced in passing. For example in the second chapter on 'Number', Exercise 11:

> 'In the statement 7 + n = 16, the letter n stands for a certain number. As 7 + 9 = 16, n must stand for 8.'
>
> 'In the statement 5 + 7 - p = 2, the letter p stands for a number. 5 + 7 = 12 and so 12 - p = 2. p stands for 10'.

These examples are followed by questions in which the child is asked to solve simple equations involving addition and subtraction of single letters as above.

In work on angles, letters are used to mark unknown angles in questions such as:



'1. What is b if a is 40°, 71°, 61°, 10°, $23\frac{1}{2}^\circ$?' (Exercise 24)

'15.

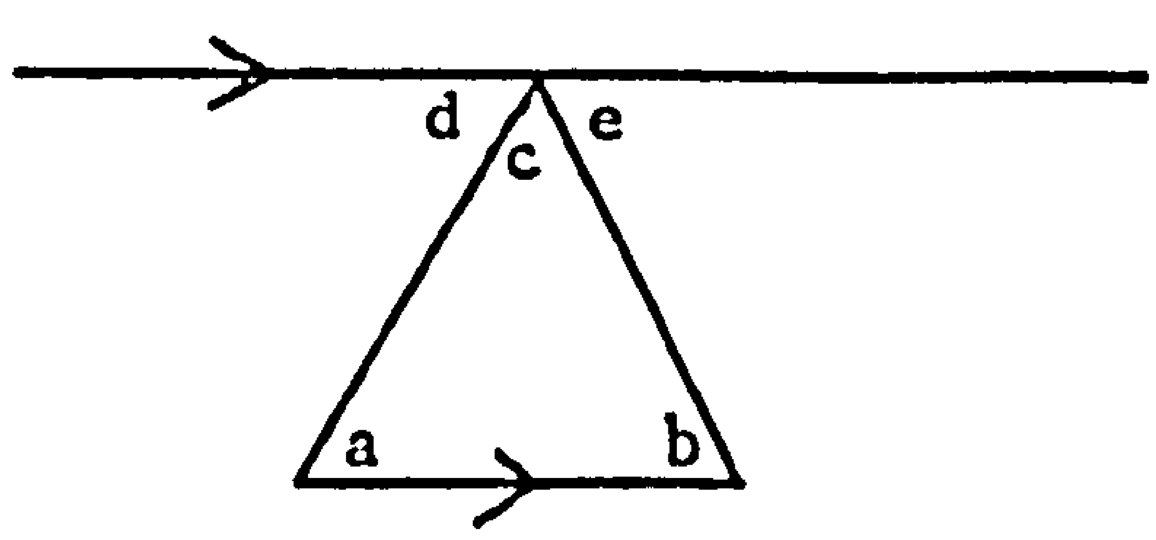


Figure 35

In Figure 35, if a = 44° and b = 59°, calculate c. Give reasons.' (Exercise 51)

In work on volumes and areas (chapter 10), formulae are introduced:

> 'We see that
>
> Volume of a cuboid = Length x Breadth x Height
>
> Formula: V = lbh'

In chapter 11 on 'Long Multiplication and Long Division', letters are again used.

'Work out 142857 x 3, 142857 x 2 and 142857 x 6. What do you notice about the answers? If 142857 x a ends in 8, what is a? Do the multiplication.

If 142857 x b ends in 5, what is b?

What happens for 142857 x 8 and 142857 x 9?' (Exercise 55)

The algebra chapter (chapter 17) introduces letters by using the commutativity of multiplication as an example where a general statement would be useful ('m x n = n x m, where m and n are any two numbers'), and goes on to stress the non-commutativity of division. It also expresses the rule for the sum of angles on a straight line as:

'p + q = 180'

and asks for various substitutions into this:

'If p = 60, what is q?'

It goes on from this to use a table of specific values of time and distance travelled for an aircraft in order to derive general statements:

'In 5 hours, the aircraft travels 620 x 5 = 3100 km'

All such statements are contained in the one statement:

'In t hours the aircraft travels 620 x t kilometres.'

In this introductory section to the chapter, the text has used algebra in a number of ways and to perform a number of functions.

The exercise which follows the introductory section asks for generalised statements similar to that quoted above. The text then goes on to formally introduce the ideas that 'a+a+a+a+a = 5 x a = 5a' and also '5a + 3a = 8a'. There follows an exercise based on these ideas asking the child to 'Write in shorter form'.

The next section uses two or more variables and similar statements which need to be abbreviated; the idea of substitution is also introduced. Following this are sections on multiplication and the use of indices, and the chapter concludes with some miscellaneous exercises.

'Headway 2'

'Headway 2' introduces algebra from the number puzzles point of view. This is followed by the solution of simple equations involving one operation.

Examples of Exercises Used

Set 1

1. How many minutes are there in:

(a) 2 hours (6) 5 hours (c) n hours?

Set 2

Write in shorter form:

1. $b + b + b + b$

2. $n \times 3$

3. $5e + 4e$

Set 3

Write in shorter form:

1. $a + a + m + m + m$

2. $3x + 2a + 5x$

3. $4m + 9y - 3y$

Set 4

Write in simplest form:

1. $3 \times t \times 4$

2. $3a \times 4b \times 5$

3. When $a = 2$, $b = 5$, $c = 3$, find:

(a) $a + c$ (b) abc (c) $3b + 2c$

Set 5

Simplify where possible:

1. $y + y + y$

2. $a \times a \times a \times a$

3. $n \times 4$

4. $2a + 5a$

5. $3a + 2b$

6. $6 \times 5a$

7. A traingle has sides of n, m and p cm. State the perimeter.

8. A number y is multiplied by 3 and then 4 is added. Write down an expression for the result.

Set 6

Write algebra equations for these sentences:

1. I think of a number, call it m, I add 4, my answer is 11.

Set 7

Find the value of the letter:

1. $a + 4 = 7$

2. $y - 3 = 9$

3. $5b = 15$

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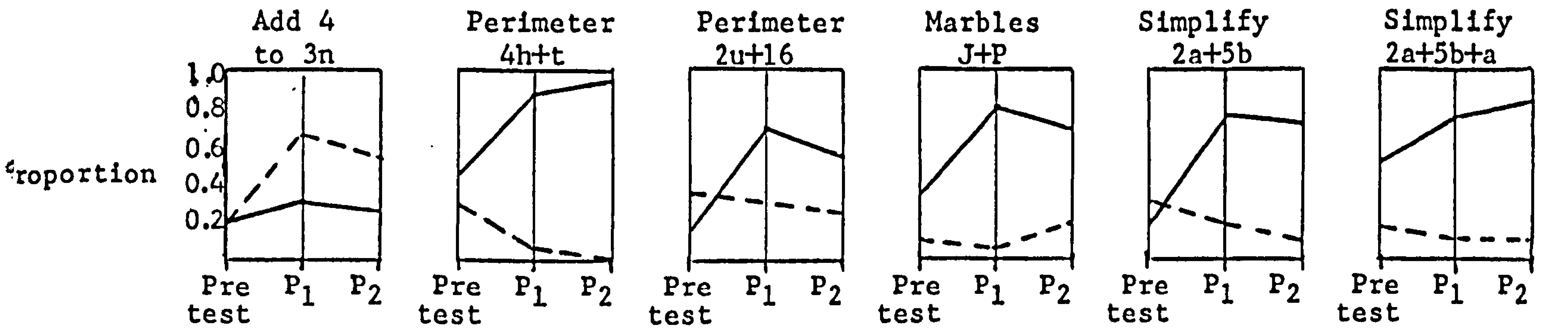
AVAILABLE

Some text bound close to the spine.

APPENDIX 12

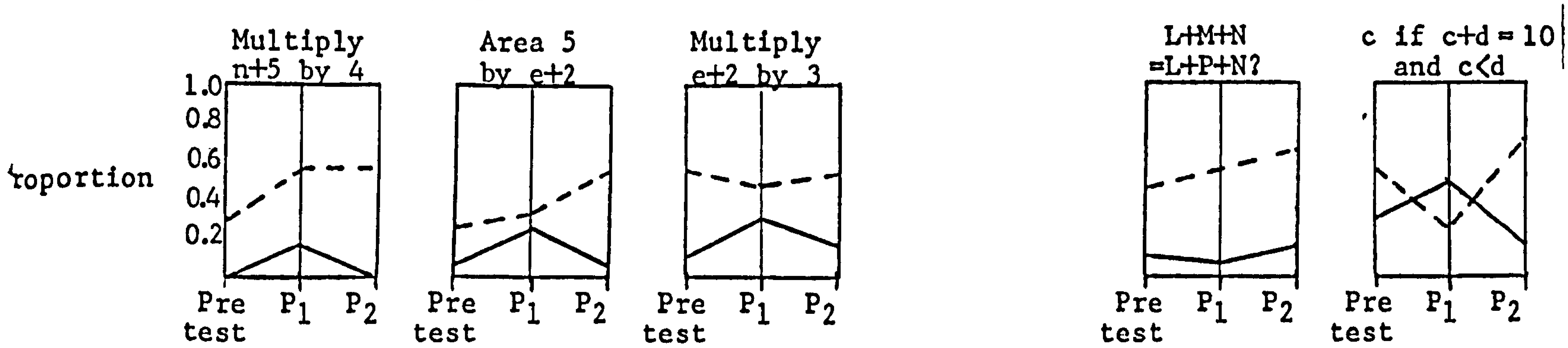
The following diagrams illustrate the changes in performance of children in schools C, D and E of the class teaching (other teachers) study on individual items from the pre- and posttests. The solid lines in each diagram represent changes in the proportion of correct answers, and the broken lines represent changes in the proportion of the 'error' answers under study.

Items: Conjoining

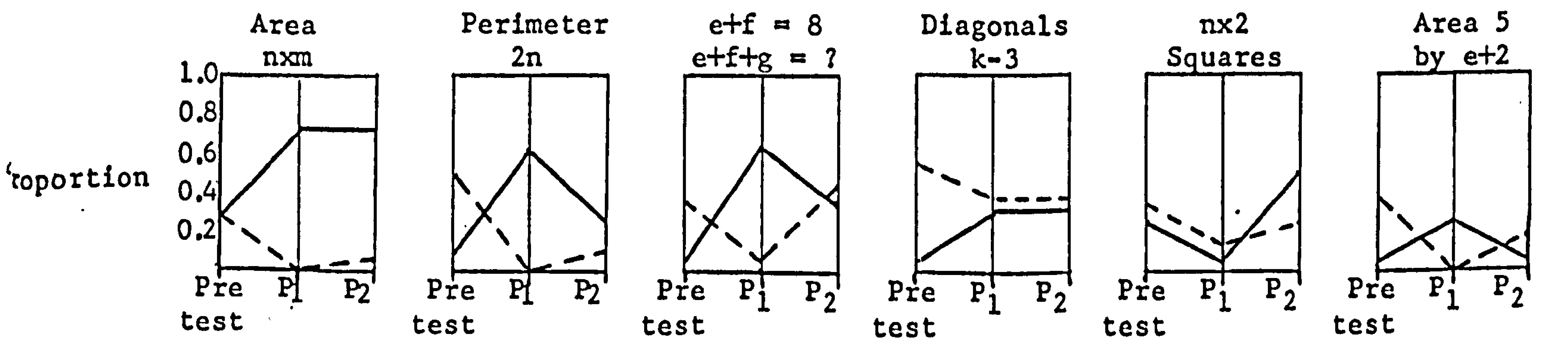


Items: Use of Brackets

Letter Interpretation

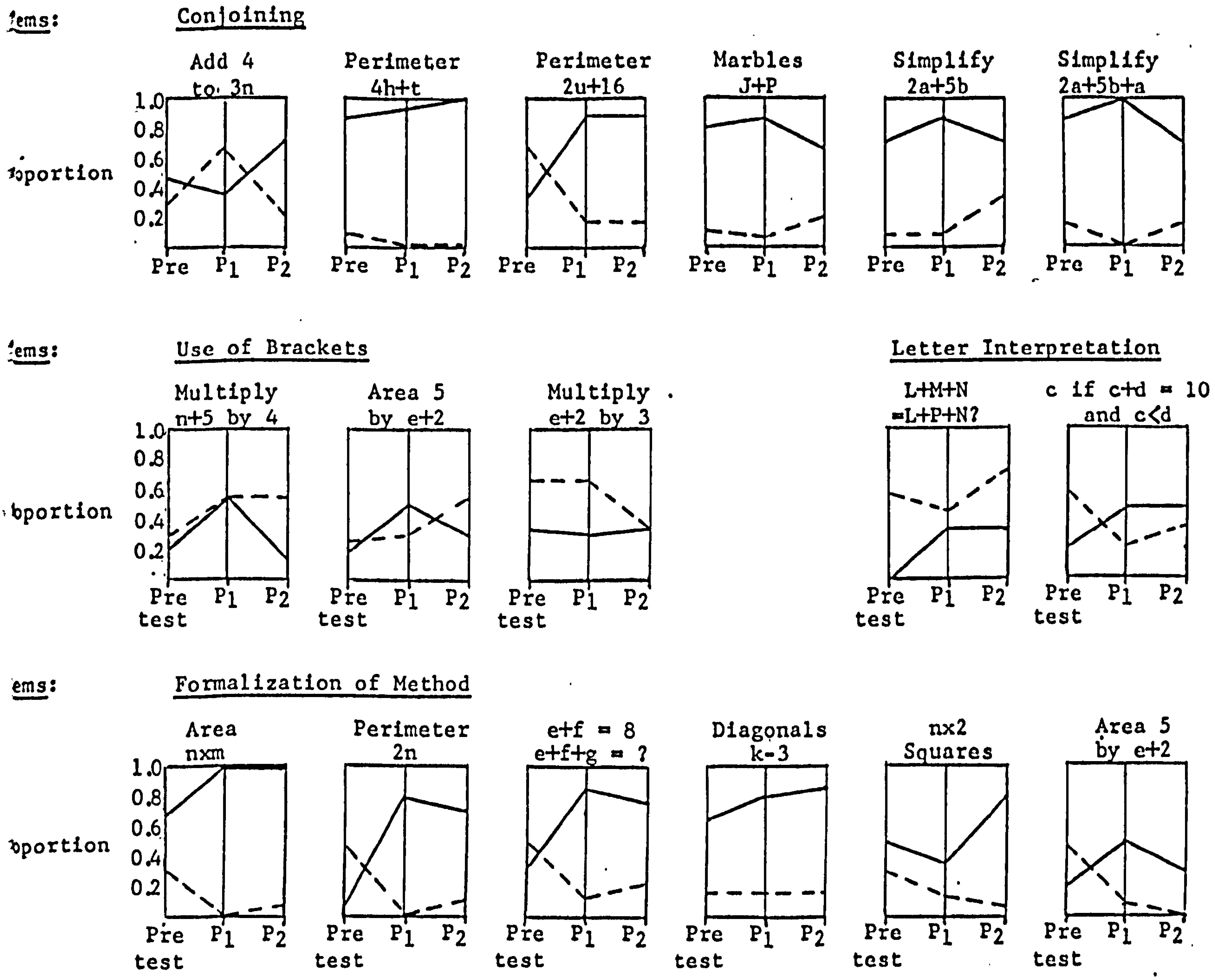


Items: Formalization of Method

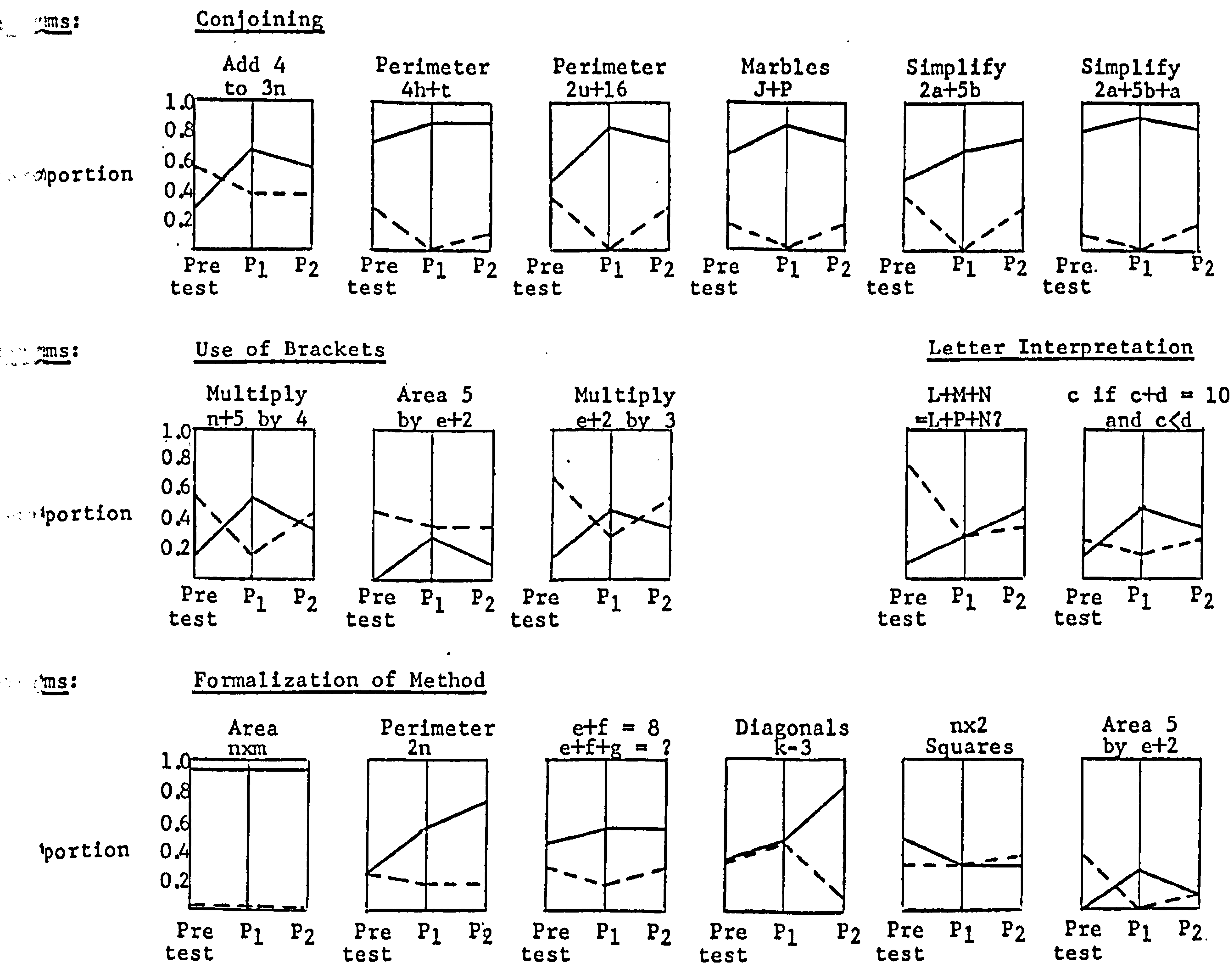


School C (3rd year, age 14), n=16

P_1 = immediate posttest, P_2 = delayed posttest



School D (4th year, age 15), n=14
P_1 = immediate posttest, P_2 = delayed posttest



School E (4th year, age 15), n=11

P_1 = immediate posttest, P_2 = delayed posttest

APPENDIX 13

Some Related Publications

The crux of the matter is that for all points P, the area of the spherical "cap" of the earth bounded by the circle of latitude on which P lies must equal the area of the circle of radius NQ on the map, where Q is the point onto which P projects. The cap is indicated by the shaded area of Figure 6; clearly because of the curvature of the earth, the radius XP of its bounding circle is less than NQ.

The area of the cap is impossible to find without the use of the integral calculus, or some method such as the Greeks used which effectively assumes the methods of the integral calculus. The result is, however, comparatively well known and rather surprising. It states that the area of the cap is equal to the area of the curved surface of the cylinder of radius R and height XN. Thus the area of the cap is

$$2\pi R . XN$$

Mathematical Association.) Now

$$XN = R(1-\sin\theta)$$

Hence

$$\pi NQ^2 = 2\pi R^2(1-\sin\theta)$$

and

$$NQ = R\sqrt{2(1-\sin\theta)}$$

This type of projection is of great importance and should be used whenever symbols are used to indicate the density of a feature on the earth's surface. If this is not done, a very misleading impression of the relative densities in different parts of the map may be given.

Getting the answer wrong

by Lesley R. Booth, SESM Project, Chelsea College

Teachers are quite used to the idea that children sometimes get mathematics questions wrong. Sometimes the wrong answer is due to a "careless" mistake which is easily corrected; often however, the teacher feels that the error is symptomatic of some misunderstanding on the part of the child. If only we could find out *why* the child had made that error, we might be able to help the child restructure his or her concept of the problem, and so avoid the error. Unfortunately, teachers usually don't have time to diagnose every child's mistakes in this way. Nor, of course, do research workers, but they may have a chance of studying "common errors" — particular wrong answers which are made by large numbers of children.

This is the task which the Strategies and Errors in Secondary Mathematics (SESM) Project[1] has set itself. Funded by the Social Science Research Council and based at Chelsea College, this project aims to investigate particular mathematical errors identified by the earlier Concepts in Secondary Mathematics and Science (CSMS) Project[2,3,4,5]. The errors chosen for study were those which were made by a large proportion (in some cases 50% or more) of the children tested by CSMS on paper-and-pencil tests of understanding in different topic areas, such as ratio, generalised arithmetic, measurement, fractions, graphs and so on. Examples of the kinds of wrong answer which were found are shown in Figure 1.

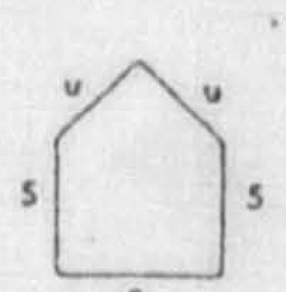
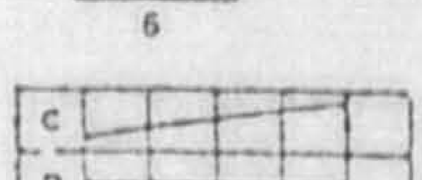
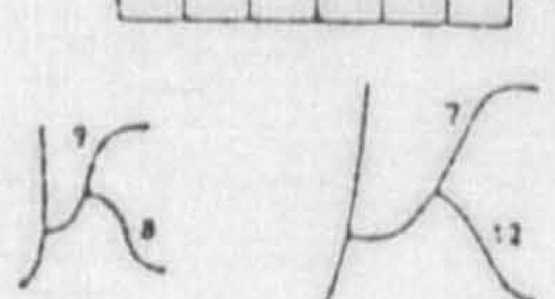
CSMS item (abridged)	"Error" answer	Percentage giving answer (13 year olds)
$\frac{2}{7}+\frac{3}{4}$	$\frac{5}{11}$	30
Add 4 onto $3n$	$3n4$, $7n$	45
Perimeter 	$2u556$, $2u16$	46
	C and D are same length	48
	13	44

Fig. 1

By carefully examining these wrong answers, and the responses which children had given when asked about their answers in interviews conducted as part of the CSMS test development programme, it was possible to suggest reasons why these particular errors may have occurred.

In essence it was suggested that many of these errors might be due to children's use of intuitive "child-methods" which are perfectly adequate for handling "easy" problems, but which do not generalise to harder questions, where success really requires the use of the "proper" mathematical methods taught in the classroom[6,7]. In addition, of course, it was thought highly likely that there would be specific kinds of misconception associated with the different areas of mathematics — algebra[8], number[9,10], fractions[11], and so on. Consequently, the SESM Project set out to examine these ideas by interviewing individual children on relevant problems, in order to discover precisely which kinds of misunderstanding contribute most to the particular errors under study.

While the results of this (still continuing) examination seem to support the notion that children often do have their own methods which break down when the questions get harder, the investigation into children's work on beginning algebra, or "generalised arithmetic", has also revealed another source of error. It seems that in this topic at least, *some* of the mistakes which children make are due not to their misinterpretation of the question or their ideas of what letters mean, nor to the methods which they use to solve the problem, but rather to their misconceptions concerning the way in which the answer

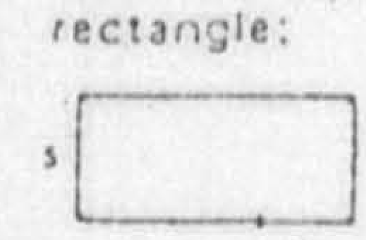
CSMS item	"Error" answer	Percentage giving answer (13 year olds)
1. What can you write for the area of this rectangle: 	$5e2$, $e10$, $10e$, $e+10$	42

Fig. 2

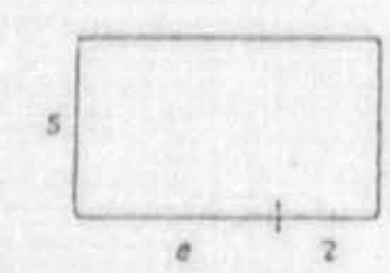
should be written down. For example, consider the commonly-observed wrong answers to Item 1 in Figure 2. These wrong answers could be due to the fact that the child ignored the e while he (or she) multiplied the numbers together, replacing the letter afterwards as it presumably had to be in there somewhere (a "letter interpretation" problem). Or the child could have treated the e as a "thing" which could be just collected up with all the numbers (another "letter interpretation" error). Alternatively, of course, the child may have had the wrong method, and thought that "area" meant multiplying everything together. However, it is also possible that the problem may have been correctly handled, but the answer incorrectly recorded. Thus when a sample of second-, third-, and fourth-year children from the "middle ability" mathematics groups of five schools in Greater London were identified as making the particular "generalised arithmetic" errors under study, and subsequently interviewed on the items in question, it was observed that 9 out of the 20 children interviewed on this item (Item 1, Figure 2) realised that the letter represented a number expressing part of the length of the base, and knew that the method required was to add the e and the 2 before multiplying by 5. However, they then went on to perform the addition, recording the sum as $e2$ or $2e$, which, when multiplied by the 5 became $5e2$, $e10$, or $10e$. Some children went on to observe that one could alternatively multiply the numbers first, giving $e+2\times5=e+10=e10$ or $10e$, which gave them a nice "check" on the answer obtained by the first method!

This confusion over the conventions of recording algebraic answers possibly reflects "psychological" stumbling blocks which are not adequately addressed by the teaching process. For example, children often show a strong tendency to apply a "combining" or "putting together" model of addition (which they have possibly carried with them since early primary school days) to a variety of situations, perhaps regardless of applicability. In addition, many children do not regard an algebraic expression as an "answer". For such children $e+2$ is not in its simplest form but still contains an instruction to act. This point of view is perhaps best illustrated from an earlier set of interviews (the rectangle in this case measured p by a added to m):

BR: Well, I would add a onto m, and times the answer by p.
LB: Do you know how you'd write that down?
BR: No, I wouldn't . . . It'll be . . . I'm not sure really, how you'd do the a plus m, to get the length.
LB: Right, now what did you tell me you'd do first?
BR: I'd add the a and m.
LB: Right, well write that down first.
BR: I'm not sure how you'd . . . (writes $a+m$ doubtfully).
LB: And then what would you do?
BR: Um, well, we've never really been taught algebra!
LB: You're doing all right!
BR: Yes, but I can't remember how you write the a and m down together, if you see what I mean. After a plussing . . .
LB: Oh, I see . . .
BR: It's a basic problem! (Laughs.)
LB: Oh I see, but you've put down $a+m$ . . .
BR: But I can't remember . . . It's presumably . . . I mean, times p is the question, but I don't know how you add up the a and the m, to get one thing (writes $a+m\times p$).
LB: So you're wondering how you can add the a and m together to get . . .
BR: Yes, one thing, to times by p.
LB: Well, *can* you get one thing? *Can* you?
BR: Well, you should be able to! Presumably you can (laughs), but I can't think of any way . . .

Other children presumably *can* think of a way, and for them the most sensible result of such an action (on $e+2$) would seem to be $e2$. It is interesting that these children, when given a numerical substitution value for e or when selecting one for themselves, tend to evaluate the expression as a sum, and not a product; however, they appear to do this by operating within the context of the original problem rather than by "reading" the $e2$ statement which they have just recorded. Regardless of the source of this observed misuse of algebraic convention, it seems clear that the error is consistently made. Thus 12 out of 16 children who were also given Item 2 in Figure 3 included $5+e2$ in their answers, stating that $e2$ meant e plus 2. Thirteen children were also given the items in Figure 4. Seven of the 8 children who replaced $k+3$ by $k3$ or $3k$ in Item 3 also included $3\times e2$ as an answer to Item 4, and the same 7 children also used this principle in the area problems. That children do regard this "conjoining" in algebraic addition as a principle rather than producing it in a capricious or careless manner is perhaps illustrated by some of their explanations during interview. Consider the response of fourth-form student CC to Item 2 in Figure 3:

2. Which of the following can you write for the area of this rectangle? Tick *every* one you think is correct:



$5\times e+2$ . . .
$5\times(e+2)$ . . .
$10e$ . . .
$5\times e2$ . . .
$5(e+2)$ . . .
$e+2\times5$ . . .
none correct . . .

Fig. 3

CC: $5\times e2$, and $5(e+2)$. That's all.
LB: Why do you say $5\times e2$?
CC: 5 times $e2$. So that means the e plus 2, the e and 2 put together, times 5. That's what the answer should be, 5 times $e2$.
LB: What does $e2$ mean?
CC: The answer to e add 2 . . . 5 times $e2$ . . . the $e2$ means you have to add the e and 2 before. So it's . . . I think I'll just stick to that one ($5\times e2$) actually, because you've got to add the e and 2 first.
LB: And $e2$ tells you to do that?
CC: Yes, to get the answer.
LB: Do any of the others tell you to do that?
CC: Just $5\times e2$ and $5(e+2)$.

Whether children use the same "principle" in reverse and interpret terms such as $e2$ and $2e$ as a sum rather than a product has not, of course, been demonstrated by this example, though one might predict at least some confusion in interpreting the "hidden" product. This was certainly so in the case of fourth-form student WA when asked about the meaning of y in the term $5y$.

WA: y could be a number, it could be a 4, making that ($5y$) 54. Or it could be 5 to the power 4, making it 20 (writes 54, 5^4).
LB: How would you know which one it was, out of those two?
WA: (Pause.) I don't know!
LB: Do you think it *could* be either, or do you think it's one of those only you're not sure which?
WA: No, it could be either, you can't really say.
LB: So it's either . . . 5 with a little 4, or it's . . . read the other one out.
WA: What . . . five four . . . no, *fifty* four!
LB: Fifty-four?
WA: Yes.
LB: Could y be anything else, besides 4?
WA: Yes — 7, 8 anything!
LB: So y could be any number? (WA nods.) Suppose I made it 23. What would you write down then?

3. Multiply $x + 3$ by 4.

4. Which of the following can you write for $e + 2$ multiplied by 3? Tick *every* one you think is correct:

$e + 6$	. . .
$3 \times (e + 2)$	. . .
$3 \times e2$	. . .
$3(e + 2)$	. . .
$3e + 6$	. . .
$e + 2 \times 3$	. . .
none correct	. . .

Fig. 4

WA: Oh! (Laughs.) Well! (laughs) five hundred and twenty-three! But I dunno — it doesn't sound very promising! I dunno. Wait, it could be 28, 5 plus 23 . . . yes . . . (pause). There again, it could be 5 to the power 23. (Writes $5 + 23$, 5^{23}.)

Here the conjoined term is interpreted initially in a place-value sense (which itself denotes implicit addition), and then subsequently as addition in the sense of combining the internal structure of the numbers (i.e. all the "ones" which comprise the numbers). The interpretation of symbolic multiplication does not seem to be the one which most immediately comes to the child's mind. In this connection another interesting possibility presents itself. If a child is interpreting the conjoined term (e.g. $5y$) as an implied sum, then one is tempted to suggest that he may "read" the power term y^5 as an implied multiplication. Certainly many children do this, and in a very consistent manner, though whether for this or some other reason remains a matter for conjecture.

The implication of all this would seem to be that we need to devote more attention to the way in which algebraic operations are recorded. A too immediate omission of the multiplication sign in algebraic products may not be wise, unless determined efforts to check the child's "reading" of the term are continually made. However, the more fundamental problem would appear to concern the child's desire to actually perform an $x + y$ addition and record a final "answer" of xy. This desire seems to have a fairly firm and deep-rooted basis, and may consequently be more resistant to "mere" teaching of notation. Nevertheless, an effort to find a way of overcoming this particular difficulty may be well rewarded not only for its likely "pay-off" in terms of the child's achievement in the whole area of generalised arithmetic, but also on motivational grounds. After all, it seems rather a pity that a student who is correctly interpreting and processing a problem should get the answer wrong because of a notational misunderstanding, and while the teacher may not interpret a wrong answer as indicative of a wrong method, can the same be said of the student?

References

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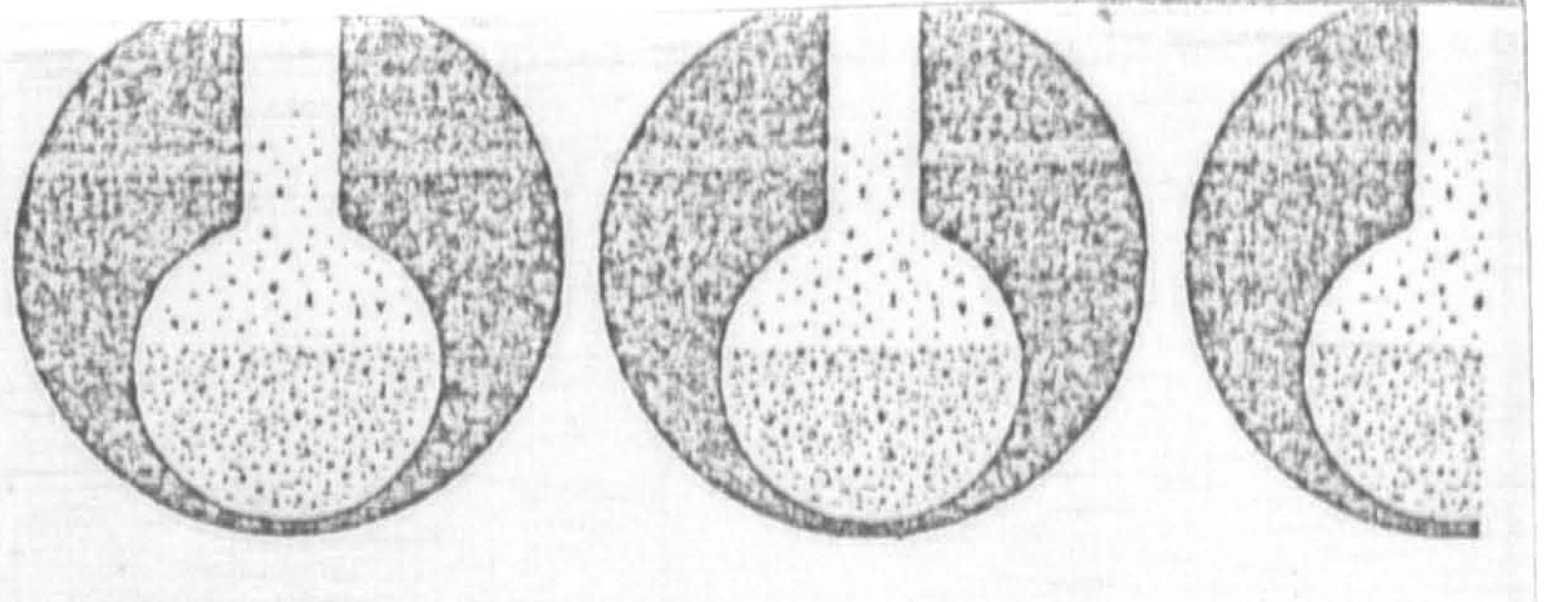
The author wishes to thank those schools which participated in the study for their kind co-operation.



Mathematics-Science Links in the Secondary School: Collaboration between Mathematics and Science Departments

Case Study — School A by A. D. Turner

Background

School A is an 11-18 girls' comprehensive school of 700 pupils set in pleasant surroundings in a town just within commuting distance to London. The school offers examinations at 'O'- and 'A'-level GCE in the separate sciences and a Mode II Science at CSE. In mathematics, the SMP course is examined and a Mode II Mathematics at CSE. Statistics is offered, too at 'O' level.

The school is fed by 40 primary schools which causes a problem in the teaching of basic number rules; subtraction is a particular difficulty. The staff are not aware of guidelines for the teaching of mathematics or science in the primary schools and the subject advisers are seen infrequently.

The school operates mixed-ability classes for the period of compulsory schooling. The mathematics department uses the SMP Books (letter series) as the main course for the first three years of school. In science, the Scottish Integrated Science course is used for years one and two, followed by the three separate sciences in the third form. Biology, chemistry and physics are taught sequentially on a 6-week rotational basis.

The head of science is a physicist who, because other staff are available, does not teach physics. He has responsibility for the teaching of 'A'-level and lower-school mathematics as well as 'A'-level chemistry and lower-school science for CSE.

Liaison

The head of science was appointed "Co-ordinator of Mathematics and Science" in 1978 with a wide brief, which included geography under his umbrella. Some items in the brief were to make science staff aware of the mathematical language and techniques which the pupils use; to transmit to mathematics staff science and geography problems for use as extension work in mathematics; and to break down insularity between departments.

The work of co-ordination got underway in 1979, through a full inter-departmental meeting. At the end of the 1979/80 academic year the head of mathematics changed. The new occupant is in full sympathy with the goals of co-ordination but has yet to make an input into this work.

There are 12 science and mathematics staff and two of these teach both subjects.

At the present time, syllabuses have been exchanged between

Ordering Your Operations

by Lesley R. Booth, SESM Project, Chelsea College

The suggestion has been made[1, 2] that part of the difficulty which some children experience in mathematics is due to their use of intuitive "child-methods" which, while being adequate for the "easy" questions, do not generalise to the "hard" ones. One of the consequences of the intuitive approach is that you don't have to be so rigorous in what you write down, since this is always to be interpreted in terms of "common sense", or in terms of the context of the question. Consequently, in answer to the item in Figure 1, for example, it wouldn't matter if you wrote down $12 \div 3$ or $3 \div 12$, since everyone would know that you meant "the number of 3's in 12". Some children certainly see no problem in recording the operation either way, as Margaret Brown has pointed out.[3, 4, 5] An "intuitive" method is always bound up with the context of the question, so that for

A bar of chocolate can be broken into 12 squares. There are 3 squares in a row. How do you work out how many rows there are?			
$12+3$	3×4	12×3	$3-12$
$6 \div 6$	$12 \div 3$	$12-3$	$3 \div 12$

Fig. 1

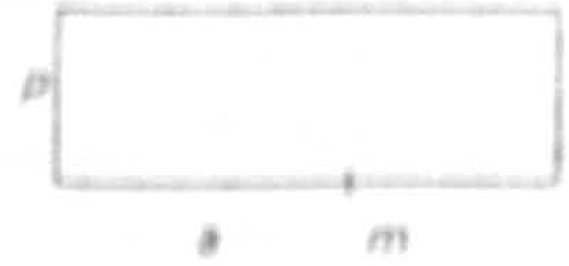
children operating in this way there is no ambiguity between $12 \div 3$ and $3 \div 12$ since the context has determined the meaning of the expression. It is only those people who are looking at the de-contextualised (formal) meaning of the two expressions who will be concerned with their difference in meaning.

This intuitive mode of approach may explain some other observations concerning what children do or do not see as necessary in mathematics. For example, such an approach also eliminates the need for such rules as "order of operations" and "use of brackets", since the obvious thing to do with $3 \times 2 + 4$, for example, is to work it out the way it comes[6], unless, of course, the context requires that the addition be done first, in which case the expression means $3 \times (2+4)$ and not $(3 \times 2)+4$. However, where the child is concerned, there is no need to actually use brackets to indicate which interpretation is intended, since for the child there is no ambiguity: the meaning is defined by the context. In most mathematical problems this view of ordering operations goes unnoticed, since the children perform the operations in the order required by the question, regardless of the order in which they write them. It is only in exercises specifically addressed to the order of operations that such a mismatch may become apparent.

Since children can handle many mathematical problems without worrying about the conventions for ordering operations or using brackets, one might ask how important a knowledge of these conventions is. However, while arithmetic problems *may* be unambiguously dealt with without reference to these rules, the same is not necessarily true of algebraic problems. Since algebraic operations such as $x+y$ cannot be "performed" in the sense of being replaced by a single value, any need to record a sequence of operations immediately requires consideration of the order conventions and the rules governing the use of brackets. If the child has not seen the need for such notions in the case of arithmetic problems, it is perhaps unlikely that he will concern himself with them in generalised arithmetic.

This possibility became apparent during the course of an investigation into the mistakes children make in beginning algebra, or "generalised arithmetic". The investigation itself is part of the larger Strategies and Errors in Secondary Mathematics Project which is funded by the Social Science Research Council and based at Chelsea College.[7] Forty-eight children from the "middle" mathematics streams in the second, third and fourth forms of four Outer London Schools were interviewed on a series of generalised arithmetic items which

1. What can you write for the area of this rectangle:



2. Multiply $k+2$ by 3

Fig. 2

included two requiring the use of brackets (Figure 2). It was noted that while just over one-third of the children who received each item were able to correctly state the method and solution, only one child realised that it was necessary to use brackets in recording the answer. The remaining children recorded their answers as $a+m\times p$ or $p\times a+m$ and $3\times k+2$ or $k+2\times 3$ respectively, even though they stated that each problem required the addition to be performed first. These children saw no ambiguity in the statements they had written: regardless of which form of expression was used, the interpreted order was for the addition to be performed first. One fourth-year student who was asked to justify his $p\times a+m$ recording did so quite happily. His reply is characteristic of other students asked:

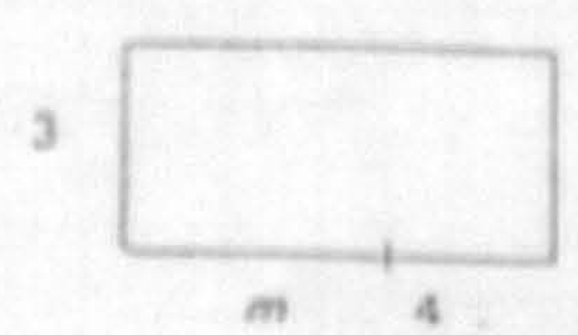
NM: p times . . . a plus m (writes $p\times a+m$).
LB: Right, so you've written down $p\times a+m$. And what would you actually do, what would you do first?
NM: I'm not with you.
LB: Right. Why did you say $p\times a+m$?
NM: Because you're timesing that side (p) by that side ($a+m$), and that side you can't do, so you've got to add that (a) onto that (m) to times it by the other side.
LB: Right so which bit would you do first?
NM: . . . I'd add those two up . . . and then I'd times it by p.
LB: And is that what you've written?
NM: Yes.
LB: Suppose I said, I thought that ($p\times a+m$) meant p times a. And then add m.
NM: Oh no, it can't be that. If you did p times a, you'd only get a bit of it. You've got to do the a plus m to get the whole length, and *then* times p. You've got to add a and m first.

And from a student who wrote $a+m\times p$, and then wrote in brackets round the $a+m$ after questioning:

LB: What have you done now?
MB: I've just put brackets round to show it's the answer (to $a+m$) you multiply by p. *But you'd know that anyway.*

Six months after these interviews, 17 of the 48 children were re-interviewed, together with seven more children from a fifth school new to the investigation. Among the interview items were the four illustrated in Figure 3. There was virtually no change in response shown by those children given items 1 and/or 2 from the earlier set and 3 and/or 4 from the second set: The only exceptions were two children who found the area

3. What can you write for the area of this rectangle?



4. Multiply $k+3$ by 4.

5. Which of the following can you write for the area of this rectangle? Tick *every* one you think is correct:

$5\times e+2$	....
$5\times(e+2)$	....
$10e$	....
$5\times e2$	....
$5(e+2)$	....
$e+2\times 5$	....
None correct	....

6. Which of the following can you write for $e+2$ multiplied by 3? Tick *every* one you think is correct:

$e+6$	....
$3\times(e+2)$	....
$3\times e2$	....
$3(e+2)$	....
$3e+6$	....
$e+2\times 3$	....
None correct	....

Fig. 3

in item 3 by dividing the rectangle into two parts, thereby successfully avoiding the use-of-brackets issue. Items 5 and 6 were included to check on the possibility that children were merely "forgetting" to write the brackets in. Faced with the choice between expressions which did and did not include brackets, it was thought that any "careless" omission of brackets would be rectified. The items were also included to give some information on the kinds of expression that children saw as equivalent.

The results of this second study supported those of the first. Thus of the 20 children given item 3, none used brackets though 11 were able to explain the method correctly, seven recording their answer as $3\times m+4$ or $m+4\times 3$, and four going on to "replace" the $m+4$ by $4m$ or $m4$. Of the 16 who were also given item 5, only one student selected the bracket expressions and nothing else; nine included at least one bracket expression among other answers, but seven of these did not recognise the equivalence of $5(e+2)$ and $5\times(e+2)$; and six students excluded the bracket expressions from their answers altogether. Similar results were obtained for items 4 and 6, though only 13 children received these questions. Those children who included both expressions with brackets and ones without saw them as equivalent expressions in which "you can put the brackets in if you want, it's just the same". The children who excluded the bracket-statements tended to do so because they didn't know what the brackets meant, or because they considered their presence unnecessary.

Thus it would seem that many children have developed for themselves a general "law" concerning the order of operations (even if they do not state it in so many words!): "an expression such as $2\times 3+4$ or $4+3\times 2$ is to be evaluated from left to right *unless* the context dictates that the addition (or multiplication) is to be performed first, in which case the two expressions are equivalent and mean add (or multiply) first". The importance of context for meaning or understanding has been discussed in the psychological/philosophical literature.[8, 9] However, the apparent ease with which children may change their interpretation of the same arithmetic statement according to the demands of the context and with no recognition of any corresponding ambiguity, is perhaps not widely appreciated. Certainly it would seem that as long as children do not recognise any ambiguity in the way operations are ordered they will also not recognise any need for rules governing this order, nor for any requirement for the use of brackets. And while children are approaching mathematical problems in an intuitive way, it is unlikely that they will recognise the ambiguity. Consequently, if we do in fact require children to have an understanding of these conventions (and for those children who are going to be taught algebra such conventions may be essential), then it would seem that we must first address ourselves to the problem of creating an awareness of the need for such conventions, and hence to the whole issue of the "intuitive" status of many children's mode of mathematical approach.

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SUMS AND BRACKETS

by **Lesley R. Booth,** Chelsea College

Three recent articles in *Mathematics in School* looked at children's ability to choose the appropriate number operation to model a given problem[1] and the degree to which children work effectively with two particular examples of mathematical notation and convention, namely using brackets and expressing algebraic sums[2,3].

It was thought useful to examine aspects of this work further and to gain more evidence concerning the scale of the problem (the work on notation and convention had involved only a small number of children in individual interview). Four teachers from four schools volunteered their assistance, and the results of the investigation are summarised below.

Choosing the Right Operation

126 children aged 12 to 16 years were given items from the CSMS "Number Operations" test[4] and the CSMS test on "Place Value and Decimals" of the kind shown in Figure 1, where children are asked to select the operation(s) needed to solve each problem. Of the 126 children tested, only five gave correct answers to every item, and 105 children had two or more items wrong (Figure 2).

Subsequent work with the children indicated that the errors made on this kind of task are due mainly to the following factors:

(a) Some children solve these kinds of problem by using their own "child-methods"[5,6]: intuitive processes which they do not symbolise mathematically. These children do not solve the problems by operating "formally" with mathematics: they do not consider the structure of the problem and then select the appropriate mathematical operation. When asked to operate in such a fashion, by choosing the correct "sum" to match a problem, they are consequently unable to do so.

(b) Other children can select the correct operation but believe that all operations are commutative. Consequently they regard such expressions as $391 \div 17$ and $17 \div 391$ or $6.44 - 8.37$ and $8.37 - 6.44$ as equivalent, and will choose either (or both).

(c) many children adhere to the rule that "you always divide the large number by the small one (or subtract the small from the large number)". Reference to the context of most of the problems they meet supports this rule. Consequently the order in which the operation is written is irrelevant, since it will always be performed in accordance with the above rule.

(d) Some children interpret $6 \div 72$ as "6 divided into 72", perhaps confounding the expression with the $6\overline{)72}$ form of recording division. "$4 - 17$" is similarly translated as "take 4 away from 17".

Brackets and Algebraic Addition

991 children aged 13 to 16 years were given a series of items designed to test their recognition of the need to use brackets in algebraic statements, and the degree to which children view expressions such as $e+2$ and $e2$ as equivalent. Two of the items used are shown in Figure 3. A large proportion of children in each year group appeared to regard brackets as superfluous in that they considered expressions with and without brackets to be equivalent. In addition, one-third of third-year and one-fifth of fourth-year students equated expressions such as $e2$ or yz with $e+2$ and $y+z$ (Figure 4). The equivalence of expressions with and without brackets is not only allowed by the "weaker" mathematics students. KL is a fourth-year student who is top of her class in a girls' selective high school (not used in the above sample):

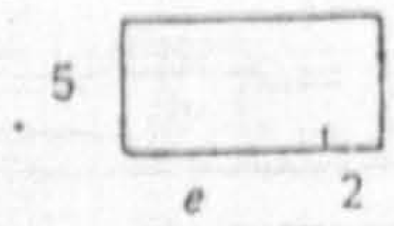
A gardener has 391 daffodils. These are to be planted in 23 flowerbeds. Each flowerbed is to have the same number of daffodils. How do you work out how many daffodils will be planted in each flowerbed?	$391 - 23$ $23 - 391$ $391 + 23$ 23×17	$23 \div 391$ 391×23 $23 + 23$ $391 \div 23$
My car can go on 41.8 miles on each gallon of petrol on a motorway. How many miles can I expect to travel on 8.37 gallons?	$41.8 + 8.37$ $41.8 \div 8.37$ $41.8 - 8.37$	$8.37 \div 41.8$ $8.37 - 41.8$ 8.37×41.8

Fig. 1

Year	Level	Test	No. of Items	All Correct	2 or More Wrong	Number Tested
1st	Mixed	Number	9	1	27	30
2nd	Top	Decimals	6	1	23	32
3rd	Low					
	Middle	Number	9	0	22	22
4th	CSE	Decimals	6	3	22	30
5th	CSE	Number	9	0	11	12
	Arith.					
		TOTAL:		5	105	126

Fig. 2

1. Which of the following can you write for the area of this rectangle? Tick *every* one you think is correct:



$5 \times e + 2$	......
$5 \times (e + 2)$	......
$10e$	......
$5 \times e\,2$	......
$5(e + 2)$	......
$e + 2 \times 5$	......
None correct	......

2. What does yz mean? Tick *every* answer you think is correct:

y and z	......
$y \times z$	......
$y + z$	......
$25 + 26$	......
25×26	......

Other answer (please write) ....

Fig. 3

Expressions for area 5 by $e+2$:	2nd Yr	3rd Yr	4th Yr	5th Yr
Correct	6(8%)	29(10%)	80(20%)	82(34%)
Brackets and non-brackets equivalent	34(43%)	105(37%)	184(46%)	92(38%)
Brackets excluded	20(25%)	77(27%)	63(16%)	37(15%)
$5 \times e2$ included	29(37%)	90(31%)	75(19%)	38(16%)
Meaning of yz:				
Ans. including $y+z$ and/or $25+26$	31(39%)	100(35%)	73(18%)	41(17%)
Number tested:	79	286	397	239

Fig. 4

KL (explaining her inclusion of the expression $e+2 \times 5$ in item 1, Figure 3): ... *it would be true but it's ... it would be more true if it had the brackets round to show you ... but I would say it was right. It's just that it's not written how it ought to be written. I'd know that someone had understood it if they put that, but that they just hadn't thought it was necessary to put the brackets round. It's correct really.*

Work with children on the use of brackets suggests that most children know about brackets but do not consider their use necessary. Two of the reasons for this were outlined in one of the articles mentioned earlier[3], namely:

(a) you perform a string of operations in the order written;
(b) a given context may require a particular operation to be done first, in which case you do this one first regardless of the way the expression is written.

To these may be added a third reason:

(c) you will in any case get the same answer regardless of the order in which you compute a sequence of operations.

Even at fourth-year level, students expressed surprise that $(3+4) \times 5$ and $3+(4 \times 5)$ did *not* produce the same answer.

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